

INVARIANT MEASURES FOR NON-PRIMITIVE TILING SUBSTITUTIONS

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ABSTRACT. We consider self-affine tiling substitutions in Euclidean space and the corresponding tiling dynamical systems. It is well-known that in the primitive case the dynamical system is uniquely ergodic. We investigate invariant measures when the substitution is not primitive, and the tiling dynamical system is non-minimal. We prove that all ergodic invariant probability measures are supported on minimal components, but there are other natural ergodic invariant measures, which are infinite. Under some mild assumptions, we completely characterize σ -finite invariant measures which are positive and finite on a cylinder set. A key step is to establish recognizability of non-periodic tilings in our setting. Examples include the “integer Sierpiński gasket and carpet” tilings. For such tilings the only invariant probability measure is supported on trivial periodic tilings, but there is a fully supported σ -finite invariant measure, which is locally finite and unique up to scaling.

1. INTRODUCTION

We consider self-affine substitution tilings of \mathbb{R}^d . A tile is a compact subset of \mathbb{R}^d which is a closure of its interior. A tiling is a set of tiles with disjoint interiors whose union is all of \mathbb{R}^d . We restrict ourselves to tilings which satisfy the (translational) *finite pattern condition*, abbreviated FPC, i.e. for any $R > 0$, there are finitely many patterns, or patches, of diameter less R , up to translation. In particular, there are finitely many tiles up to translation; we call some representatives of equivalence classes the *prototiles*. Sometimes tiles of the same shape need to be distinguished; this is achieved by considering a tile as a pair $T = (F, j)$ where $F = \text{supp}(T)$ is a compact set (the *support* of the tile), and $j \in \{1, \dots, \ell\}$ for some $\ell \geq 1$, is a *label* (or “type”, or “color”) of the tile. When translating a tile or a patch, the labels of the tiles are preserved. Let \mathcal{A} be a finite set of prototiles and \mathcal{A}^+ the set of patches whose every tile is a translate of some $A \in \mathcal{A}$. Given an expansive linear map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we say that $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ is a *tile substitution with expansion* φ if the union of tiles in $\omega(A)$ equals $\varphi(\text{supp}(A))$. In other words, every “inflated tile” can be subdivided into translates of the prototiles. This property allows us to iterate the substitution and obtain a family of patches $\omega^n(A)$, $A \in \mathcal{A}$, $n \geq 1$. The *substitution tiling space* $X_{\mathcal{A}, \omega}$ is defined as the set of all tilings of \mathbb{R}^d whose every patch is a translate of a subpatch of $\omega^n(A)$ for some $A \in \mathcal{A}$ and $n \in \mathbb{N}$. We assume that this space has the FPC property. We also assume that every prototile $A \in \mathcal{A}$ is *admissible*, that is, there exists a tiling $\mathcal{T} \in X_{\mathcal{A}, \omega}$ with $A \in \mathcal{T}$. The space $X_{\mathcal{A}, \omega}$ is compact

2000 *Mathematics Subject Classification.* Primary: 37B50; Secondary: 37A99 .

Key words and phrases. tiling systems, substitutions, invariant measures.

M. I. Cortez acknowledges financial support from proyecto Fondecyt 1100318. B. Solomyak was partially supported by NSF grant DMS-0654408.

in the usual “local” metric, in which two tilings are considered to be close if they agree on a large ball around the origin up to a small translation (see the next section for precise definitions), and \mathbb{R}^d acts on $X_{\mathcal{A}, \omega}$ continuously by translations. This is the *tiling dynamical system* associated with ω . To a tiling substitution ω we associate the *substitution matrix* M whose entry $M(i, j)$ equals the number of tiles of type i which appear in the substitution ω applied to a tile of type j . The substitution is *primitive* if $M^k > 0$ for some $k \in \mathbb{N}$. Substitution tiling dynamical systems have been studied almost exclusively in the primitive case, when they are minimal and uniquely ergodic. Here we begin the investigation of non-primitive tiling substitutions. A tiling substitution can be viewed as a generalization of a symbolic substitutions, see [20, 19, 21]. Recently, a systematic investigation of non-primitive (one-dimensional) symbolic substitutions has been started in [4, 5] (see also [10, 33, 13]). Our work builds on [5]; however, we have to introduce many new ingredients. The substitution matrix M is non-negative, and we can consider its irreducible components, see [16, 4.4]. We will show that $X_{\mathcal{A}, \omega} = X_{\mathcal{A}, \omega^k}$ for $k \in \mathbb{N}$; thus, by raising the substitution to a positive power we can assume, without loss of generality that all irreducible components are primitive or equal to [0]. Suppose that M has irreducible components M_1, \dots, M_ℓ , and let $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ be the corresponding subsets of the prototile set. By reordering the prototiles, we can assume that M has a block upper-triangular form, with the diagonal blocks M_i , $i \leq \ell$. In terms of the substitution, this means that $\omega(A)$, $A \in \mathcal{A}_j$, contains only translated of the tiles from $\bigcup_{i=1}^j \mathcal{A}_i$. Let $m \geq 1$ be the number of “minimal” irreducible components having the property that $\omega(A) \subseteq \mathcal{A}_i^+$ for all $A \in \mathcal{A}_i$. Then $\omega_i := \omega|_{\mathcal{A}_i}$ is the usual primitive substitution for $i \leq m$. It turns out that $X_i := X_{\mathcal{A}_i, \omega_i}$, $i \leq m$, are precisely the minimal components of the tiling dynamical system. Our first main result is the following.

Theorem A. *All ergodic invariant probability measures for the substitution tiling system are supported on minimal components.*

The proof uses the pointwise ergodic theorem and the fact that only the patches from minimal components may have a positive frequency in a tiling. However, this is only the beginning of the story, as there are natural and interesting *infinite* invariant measures. In order to characterize them, we need the property of *recognizability*, namely, the invertibility of the substitution map ω extended to the tiling space $X_{\mathcal{A}, \omega}$. We prove that it holds whenever the tiling substitution is non-periodic, namely $\mathcal{T} + \mathbf{v} \neq \mathcal{T}$ for $\mathcal{T} \in X_{\mathcal{A}, \omega}$ and $\mathbf{v} \neq \mathbf{0}$, generalizing [25] from the primitive case. Even more, we establish recognizability of non-periodic tilings when the tiling space does contain periodic ones, under a mild geometric condition (the “non-periodic border” condition, see Section 4 for details). Applying recognizability, we construct a sequence of *nested Kakutani-Rokhlin partitions* of the transversal which allows us to determine the natural σ -finite measures, first on the transversal, and then on the tiling space. (Actually, in general it is only a covering, but it becomes a partition when restricted to the set of non-periodic tilings, which is enough for our purposes.) The *transversal* is defined as the set of tilings which have a tile with a “puncture” at the origin; this is consistent with a view of the tiling space as a lamination. *Transverse measures* are in 1-to-1 correspondence with

invariant measures. Although the use of transverse measures in tiling dynamics is by now standard, see [2], we have to extend the theory to our setting, namely, to the non-primitive case, and to include σ -finite measures; this is done in the Appendix. In order to state our results on σ -finite measures, we need to introduce some terminology. There are irreducible components M_i and the corresponding prototile subsets \mathcal{A}_i , which we call “maximal”: they are characterized by the property

$$\omega(A) \text{ contains a tile of type } \mathcal{A}_i \implies A \in \mathcal{A}_i.$$

Let $p \geq 1$ be the number of maximal components; they correspond to prototile subsets $\mathcal{A}_{\ell-p+1}, \dots, \mathcal{A}_\ell$. Denote by Y_i , for $i \geq \ell - p + 1$, the set of tilings $\mathcal{T} \in X_{\mathcal{A}, \omega}$ which contain at least one tile of type \mathcal{A}_i . The subsets $Y_i \subseteq X_{\mathcal{A}, \omega}$ are non-empty (by the admissibility assumption), open, and invariant. We will show that (Y_i, \mathbb{R}^d) is a non-compact minimal tiling dynamical system for $i \geq \ell - p + 1$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^p Y_i$ is dense in $X_{\mathcal{A}, \omega}$. We call Y_i the *maximal* components of the tiling dynamical system. Now we can state our second main result.

Theorem B. *Suppose that the tiling substitution satisfies the non-periodic border condition (in particular, it is satisfied if the substitution is non-periodic). Then for each $i = \ell - p + 1, \dots, \ell$, there is a unique, up to scaling, ergodic invariant measure supported on Y_i , such that every point has an open neighborhood with positive finite measure.*

Remarks. 1. It is often the case that $X_{\mathcal{A}, \omega} \neq \bigcup_{i=1}^m X_i \cup \bigcup_{i=\ell-p+1}^\ell Y_i$, and there are other infinite invariant measures, but those described in Theorem B are the most natural ones. In Section 5 we classify all ergodic invariant measures which are positive and finite on a “cylinder set”; they correspond to some irreducible components of M .

2. There is, in general, a greater variety of invariant measures for (one-dimensional) symbolic substitutions than for tile substitutions considered here; in particular, there are sometimes ergodic invariant probability measures of full support, see [5]. The reason is that in our case the vector of volumes of the prototiles is always a strictly positive left eigenvector of the substitution matrix, whereas for a symbolic substitution such an eigenvector may fail to exist.

The paper is organized as follows. Section 2 contains preliminaries, including the topological results about minimal and maximal components. Theorem A is proved in Section 3. We obtain recognizability results, which are of independent interest, in Section 4, and in Section 5 we investigate σ -finite invariant measures and prove Theorem B. (We would like to point out that Sections 4 and 5 can be read independently.) Section 6 is devoted to examples and concluding remarks. For specific examples the recognizability properties can usually be checked directly. In Section 7 (the Appendix) we treat transverse measures.

Acknowledgment. We are grateful to Karl Petersen for his interest, helpful comments, and suggestion to consider the “Sierpiński carpet” and “gasket” tilings.

2. PRELIMINARIES

2.1. Tilings. Fix a set of “types” labeled by $\{1, \dots, N\}$, with $N \geq 1$. A *tile* in \mathbb{R}^d is defined as a pair $T = (F, i)$ where $F = \text{supp}(T)$ (the support of T) is a compact set in \mathbb{R}^d which is the closure of its interior, and $i = \ell(T) \in \{1, \dots, N\}$ is the type of T . A *tiling* of \mathbb{R}^d is a set \mathcal{T} of tiles such that $\mathbb{R}^d = \bigcup \{\text{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles (or rather, their supports) have disjoint interiors. A patch P is a finite set of tiles with disjoint interiors. The *support of a patch* P is defined by $\text{supp}(P) = \bigcup \{\text{supp}(T) : T \in P\}$. The *diameter of a patch* P is $\text{diam}(P) = \text{diam}(\text{supp}(P))$. The *translate* of a tile $T = (F, i)$ by a vector $\mathbf{g} \in \mathbb{R}^d$ is $T + \mathbf{g} = (F + \mathbf{g}, i)$. The translate of a patch P is $P + \mathbf{g} = \{T + \mathbf{g} : T \in P\}$. We say that two patches P_1, P_2 are *translationally equivalent* if $P_2 = P_1 + \mathbf{g}$ for some $\mathbf{g} \in \mathbb{R}^d$. Finite subsets of \mathcal{T} are called \mathcal{T} -patches. For a set $B \subseteq \mathbb{R}^d$ we write

$$[B]^{\mathcal{T}} = \{T \in \mathcal{T} : \text{supp}(T) \cap B \neq \emptyset\}.$$

Definition 2.1. We say that a tiling \mathcal{T} has (translational) finite patch complexity (FPC), or satisfies the finite pattern condition, if for any $R > 0$ there are finitely many \mathcal{T} -patches of diameter less than R up to translation equivalence. This definition naturally extends to any collection of tilings.

Definition 2.2. A tiling \mathcal{T} is repetitive if for any patch $P \subset \mathcal{T}$ there is $R > 0$ such that for any $\mathbf{x} \in \mathbb{R}^d$ there is a \mathcal{T} -patch P' such that $\text{supp}(P') \subset B_R(\mathbf{x})$ and P' is a translate of P .

2.2. Tile substitutions, self-affine tilings. A linear map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is expansive if all its eigenvalues lie outside the unit circle.

Definition 2.3. Let $\mathcal{A} = \{A_1, \dots, A_N\}$ be a finite set of tiles in \mathbb{R}^d such that for $i \neq j$ the tiles A_i and A_j are not translationally equivalent; we will call them prototiles. We assume that every prototile is “centered at the origin”, in the sense that $\mathbf{0} \in \text{int}(\text{supp}(A_j))$ for all j . Denote by \mathcal{A}^+ the set of patches made of tiles each of which is a translate of one of the prototiles. A map $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ is called a tile substitution with expansion φ if

$$(2.1) \quad \text{supp}(\omega(A_j)) = \varphi(\text{supp}(A_j)) \quad \text{for } j \leq N.$$

In plain language, every expanded prototile $\varphi(A_j)$ can be decomposed into a union of tiles (which are all translates of the prototiles) with disjoint interiors. The substitution ω is extended to all translates of prototiles by $\omega(\mathbf{x} + A_j) = \varphi(\mathbf{x}) + \omega(A_j)$, and to patches by $\omega(P) = \bigcup \{\omega(T) : T \in P\}$. This is well-defined due to (2.1). Denote by $X_{\mathcal{A}}$ the set of tilings whose tiles belong to \mathcal{A} up to translation; note that ω acts on $X_{\mathcal{A}}$ as well.

Definition 2.4. Let ω be a tile substitution. A patch P is said to be legal if there exists $n \geq 1$, $A_j \in \mathcal{A}$, and $\mathbf{x} \in \mathbb{R}^d$, such that $P + \mathbf{x} \subseteq \omega^n(A_j)$. Denote by $X_{\mathcal{A}, \omega} \subseteq X_{\mathcal{A}}$ the set of all tilings of \mathbb{R}^d whose every patch is legal. The set $X_{\mathcal{A}, \omega}$ is called the tiling space corresponding to the substitution. We say that the substitution ω has FPC if the space $X_{\mathcal{A}, \omega}$ has FPC. The substitution ω is admissible if for every prototile A_j there exists $\mathcal{T} \in X_{\mathcal{A}, \omega}$ such that $A_j \in \mathcal{T}$.

The additive group \mathbb{R}^d acts on $X_{\mathcal{A}, \omega}$ by translations; this action $(X_{\mathcal{A}, \omega}, \mathbb{R}^d)$ is called the *tiling dynamical system* or the *self-affine tiling dynamical system* associated to ω . It is clear from the definitions that $\omega(X_{\mathcal{A}, \omega}) \subseteq X_{\mathcal{A}, \omega}$.

We use a *tiling metric* on $X_{\mathcal{A}, \omega}$, in which two tilings are close if after a small translation they agree on a large ball around the origin. To make it precise, we say that two tilings $\mathcal{T}_1, \mathcal{T}_2$ agree on a set $K \subset \mathbb{R}^d$ if

$$\text{supp}(\mathcal{T}_1 \cap \mathcal{T}_2) \supseteq K.$$

For $\mathcal{T}_1, \mathcal{T}_2 \in X_{\mathcal{A}, \omega}$ let

$$\tilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2) := \inf\{r \in (0, 2^{-1/2}) : \exists \mathbf{g}, \|\mathbf{g}\| \leq r \text{ such that}$$

$$\mathcal{T}_1 - \mathbf{g} \text{ agrees with } \mathcal{T}_2 \text{ on } B_{1/r}(\mathbf{0}).\}$$

Then

$$\varrho(\mathcal{T}_1, \mathcal{T}_2) := \min\{2^{-1/2}, \tilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2)\}.$$

Theorem 2.5. [22] (see also [21]). $(X_{\mathcal{A}, \omega}, \varrho)$ is a complete metric space. It is compact, whenever the space has FPC. The action of \mathbb{R}^d by translations on $X_{\mathcal{A}, \omega}$, given by $\mathbf{g}(\mathcal{S}) = \mathcal{S} - \mathbf{g}$, is continuous.

Definition 2.6. To the tile substitution ω we associate its $N \times N$ substitution matrix $M = M_\omega$, where $M(i, j)$ is the number of tiles of type A_i in the patch $\omega(A_j)$. The substitution ω is called primitive if the substitution matrix is primitive, that is, if there exists $k \in \mathbb{N}$ such that M^k has only positive entries.

Theorem 2.7. [12] (see also [21, Sec. 5]).

- (i) An FPC tiling system is repetitive if and only if it is minimal, that is, every orbit $\{\mathcal{S} - \mathbf{g} : \mathbf{g} \in \mathbb{R}^d\}$ is dense in X .
- (ii) An FPC substitution tiling system is minimal if and only if the substitution is primitive.

So far, much of the theory has focused on primitive tile substitutions ω . In this paper we investigate what happens in the absence of primitivity and repetitivity, in the context of tiling systems which satisfy FPC.

Lemma 2.8. Let ω be an admissible substitution on the set of prototiles \mathcal{A} . Then $\omega : X_{\mathcal{A}, \omega} \rightarrow X_{\mathcal{A}, \omega}$ is onto.

Proof. Let $\mathcal{T} \in X_{\mathcal{A}, \omega}$. For $n \geq 1$, we define

$$P_n = [B_n(\mathbf{0})]^\mathcal{T}.$$

Since the patches of \mathcal{T} are legal, there exist $k_n \geq 1$, $\mathbf{x}_n \in \mathbb{R}^d$ and $A^{(n)} \in \mathcal{A}$ such that

$$P_n + \mathbf{x}_n \subseteq \omega^{k_n}(A^{(n)}).$$

Because $|\mathcal{A}| < \infty$, there exist an infinite subset $I \subseteq \mathbb{N}$ and $A \in \mathcal{A}$ such that

$$P_n + \mathbf{x}_n \subseteq \omega^{k_n}(A) \text{ for every } n \in I.$$

Taking subsequences if it is necessary, we can suppose that $k_n < k_{n+1}$. For $n \in I$, let $Q_n = \omega^{k_n-1}(A)$. Observe that $P_n + \mathbf{x}_n \subseteq \omega(Q_n)$. Let $\mathcal{T}_A \in X_{\mathcal{A}, \omega}$ be such that $A \in \mathcal{T}_A$ (the tiling \mathcal{T}_A exists by the definition of admissible substitution), and let $\mathcal{T}_n = \omega^{k_n-1}(\mathcal{T}_A)$. We have

$$Q_n = \omega^{k_n-1}(A) \subseteq \omega^{k_n-1}(\mathcal{T}_A) = \mathcal{T}_n,$$

which implies $P_n + \mathbf{x}_n \subseteq \omega(\mathcal{T}_n)$. By compactness of $X_{\mathcal{A}, \omega}$, there exist a subsequence $(n_j)_{j \geq 0}$ and $\mathcal{T}' \in X_{\mathcal{A}, \omega}$ such that

$$\lim_{j \rightarrow \infty} (\mathcal{T}_{n_j} - \varphi^{-1}(\mathbf{x}_{n_j})) = \mathcal{T}'.$$

Since for every $n \in I$,

$$P_n \subseteq \omega(\mathcal{T}_n - \varphi^{-1}(\mathbf{x}_n)),$$

we get $\omega(\mathcal{T}') = \mathcal{T}$. □

Lemma 2.9. *We have $X_{\mathcal{A}, \omega} = X_{\mathcal{A}, \omega^k}$ for $k \geq 2$.*

Proof. The inclusion \supseteq is clear. For the other inclusion, let P be a legal patch for ω . Then P occurs in some $\omega^n(A)$, $A \in \mathcal{A}$. By assumption, there is a tiling \mathcal{S} in $X_{\mathcal{A}, \omega}$ which contains a tile of type A . By Lemma 2.8, there exists a tiling $\mathcal{S}' \in X_{\mathcal{A}, \omega}$ such that $\omega^{n(k-1)}(\mathcal{S}') = \mathcal{S}$. Then a tile of type A is in some $\omega^{n(k-1)}(A')$ for some $A' \in \mathcal{A}$, hence P occurs in $\omega^{nk}(A')$ which implies that P is legal for ω^k . □

2.3. Substitution matrix. Let $M \in \mathcal{M}_{N \times N}(\mathbb{Z}_+)$. The graph $G(M)$ associated to M is the directed graph whose set of vertices is $\{1, \dots, N\}$, such that there is an edge from i to j if and only if $M(i, j) > 0$. An equivalence relation is defined on the set of vertices of $G(M)$ as follows: $i \sim j$ if and only if $j = i$ or there is a path in $G(M)$ from i to j as well as a path from j to i . The matrix is *irreducible* if and only if all the vertices of $G(M)$ are equivalent, otherwise, it is *reducible*. We call the equivalence classes of $G(M)$ the *irreducible components*, see [16, 4.4].

We say that an equivalence class α *has access* to the equivalence class β , or that β is accessible from α , if and only if $\alpha = \beta$ or if there exists a path in $G(M)$ from a vertex in α to a vertex in β . This relation is denoted by $\alpha \succeq \beta$. In a similar way we say that the vertex i has access to β if there is a path in $G(M)$ from i to a vertex in β .

For an equivalence class α we denote by M_α the irreducible submatrix (diagonal block) of M corresponding to the restriction of M to α .

Now we return to our tiling substitution ω and let $M = M_\omega$ be the substitution matrix. We identify the vertex set of the graph $G(M)$ with the prototile set \mathcal{A} . Since the tiling dynamical system is completely determined by the space $X_{\mathcal{A}, \omega}$, we can, in view of Lemma 2.9, replace ω by ω^k for $k \geq 2$. The substitution matrix for ω^k is clearly M^k . By raising a reducible non-negative matrix to a power we can get rid of the “cyclic structure” of the irreducible components (this will increase the number of irreducible components if there is nontrivial

cyclic structure), see [16, 4.5] for details. Thus, we can (and will) assume, without loss of generality, that

$$(2.2) \quad \text{every irreducible block of } M \text{ is either primitive or equals } [0].$$

2.4. Minimal and maximal components. Consider the irreducible components of the graph $G(M)$ which are *maximal* in the partial order \succeq . In other words, a component α is maximal if it is not accessible from any other component. Denote by $\mathcal{A}_1, \dots, \mathcal{A}_m$ the subsets of the prototile set \mathcal{A} corresponding to these maximal components. Observe that $m \geq 1$. By the definition of the substitution matrix, we have $\omega(\mathcal{A}_i) \subseteq \mathcal{A}_i^+$, so that $\omega_i := \omega|_{\mathcal{A}_i}$ is a tile substitution on the prototile set \mathcal{A}_i . By assumption (2.2), this substitution is primitive, so $(X_{\mathcal{A}_i, \omega_i}, \mathbb{R}^d)$ is a minimal dynamical system by Theorem 2.7. Let $X_i = X_{\mathcal{A}_i, \omega_i}$ for $i \leq m$. In the next lemma we show that these are precisely the minimal components of the tiling dynamical system. (It seems counter-intuitive that minimal components correspond to maximal irreducible components of the graph, but this is just the consequence of definitions. One can also consider the graph $G(M^T)$ for the transpose of the substitution matrix; this reverses the direction of edges, so the minimal components of the dynamical system correspond to minimal irreducible components of $G(M^T)$.)

Lemma 2.10. *Suppose that ω is an admissible FPC tile substitution satisfying (2.2). Then*

- (i) *the minimal components of the tiling dynamical system $(X_{\mathcal{A}, \omega}, \mathbb{R}^d)$ are (X_i, \mathbb{R}^d) , for $i = 1, \dots, m$; they satisfy $\omega(X_i) \subseteq X_i$;*
- (ii) *for any tiling $\mathcal{T} \in X_{\mathcal{A}, \omega}$ and a prototile $A \in \mathcal{A}_j$ which occurs in \mathcal{T} , the orbit closure $\text{Clos}\{\mathcal{T} - \mathbf{g} : \mathbf{g} \in \mathbb{R}^d\}$ contains every minimal component X_i such that \mathcal{A}_j is accessible from \mathcal{A}_i .*

Proof. We already know that (X_i, \mathbb{R}^d) is minimal and $\omega(X_i) \subseteq X_i$. Part (ii) will imply that there are no other minimal components; thus, it remains to prove (ii). Let $A \in \mathcal{A}_j$ and $A + \mathbf{x} \in \mathcal{T}$ for some $\mathbf{x} \in \mathbb{R}^d$ and $j \geq 1$. By Lemma 2.8, there exists a sequence $k_n \uparrow \infty$ and prototiles $A^{(n)}$ such that

$$A + \mathbf{x} \in \omega^{k_n}(A^{(n)}) + \mathbf{x}_n \subseteq \mathcal{T} \text{ for some } \mathbf{x}_n \in \mathbb{R}^d.$$

Passing to a subsequence, we can assume that $A^{(n)} = B$ for all n . Let $1 \leq i \leq m$ be such that \mathcal{A}_j is accessible from \mathcal{A}_i , then B is accessible from \mathcal{A}_i , hence $\omega^s(B)$ contains a prototile of type \mathcal{A}_i for some $s \in \mathbb{N}$. Since \mathcal{T} contains a translates of $\omega^{n+s}(B)$ for arbitrarily large n , the closure of the orbit of \mathcal{T} contains $X_{\mathcal{A}, \omega} = X_i$. \square

Corollary 2.11. *There are at most $|\mathcal{A}|$ minimal components in $X_{\mathcal{A}, \omega}$.*

Suppose that the matrix M , and the graph $G(M)$, have ℓ irreducible components. It is also useful to consider the *minimal* irreducible components of the graph $G(M)$ (or maximal irreducible components of $G(M^T)$). The corresponding subsets \mathcal{A}_j of the prototile set are characterized by the property that for any $i \neq j$ and $A \in \mathcal{A}_i$, the substitution $\omega(A)$ does not contain tiles of type \mathcal{A}_j . Suppose there are p such components; the corresponding prototile sets are $\mathcal{A}_{\ell-p+1}, \dots, \mathcal{A}_\ell$. Note that if $j \geq \ell - p + 1$ and $A \in \mathcal{A}_j$, then the substitution $\omega(A)$

necessarily contains a tile of type \mathcal{A}_j , since otherwise the substitution is not admissible. Thus, the corresponding matrices M_j are nonzero and hence primitive by (2.2). Let

$$Y_j := \{\mathcal{T} \in X_{\mathcal{A}, \omega} : \mathcal{T} \text{ contains a tile of type } \mathcal{A}_j\}.$$

We call Y_j , $j = \ell - p + 1, \dots, \ell$, the *maximal components* of the tiling space $X_{\mathcal{A}, \omega}$.

Lemma 2.12. *Suppose that ω is an admissible FPC tile substitution satisfying (2.2). Then*

- (i) *The subsets Y_j , $j = \ell - p + 1, \dots, \ell$, are mutually disjoint, open in $X_{\mathcal{A}, \omega}$, and invariant under the translation \mathbb{R}^d -action and the substitution \mathbb{Z}_+ -action;*
- (ii) *for any $\mathcal{T} \in Y_j$, $j = \ell - p + 1, \dots, \ell$, we have $Y_j \subseteq \text{Clos}\{\mathcal{T} - \mathbf{g} : \mathbf{g} \in \mathbb{R}^d\}$;*
- (iii) *$\bigcup_{j=\ell-p+1}^{\ell} Y_j$ is dense in $X_{\mathcal{A}, \omega}$.*

Proof. (i) This is immediate; Y_j is invariant under ω because the irreducible component M_j is non-zero.

(ii) Exactly as in the proof of Lemma 2.8, we obtain $A \in \mathcal{A}$, an infinite set $I_{\mathcal{T}}$ and $\mathbf{x}_n \in \mathbb{R}^d$ for $n \in I_{\mathcal{T}}$ such that

$$(2.3) \quad [B_n(\mathbf{0})]^{\mathcal{T}} + \mathbf{x}_n \subseteq \omega^{k_n}(A) \quad \text{for } n \in I_{\mathcal{T}}.$$

The assumption $\mathcal{T} \in Y_j$ implies that $A \in \mathcal{A}_j$. But M_j is primitive, so we can use the same $A \in \mathcal{A}_j$ for any $\mathcal{S} \in Y_j$. This implies that the orbit of \mathcal{T} contains \mathcal{S} in its closure, as desired.

(iii) Let $\mathcal{T} \in X_{\mathcal{A}, \omega}$. Again we find A and $I_{\mathcal{T}}$ satisfying (2.3). By the definition of the graph $G(M)$, the vertex (prototile) A has access to one of the maximal components \mathcal{A}_j , with $\ell - p + 1 \leq j \leq \ell$. This means $A + \mathbf{z} \in \omega^{k_0}(A')$ for some $A' \in \mathcal{A}_j$ and $k_0 \in \mathbb{N}$. Let $\mathcal{T}' \in X_{\mathcal{A}, \omega}$ be a tiling containing A' , which exists by admissibility. Then $\omega^{k_0+k_n}(\mathcal{T}') - \varphi^{k_n}(\mathbf{z}) - \mathbf{x}_n \in Y_j$, and this sequence converges to \mathcal{T} . \square

2.5. Non-negative matrices. Let $M \in \mathcal{M}_{k \times k}(\mathbb{Z}_+)$ and let α be an equivalence class of $G(M)$. Denote by ρ_α its spectral radius. The class α is *distinguished* if $\rho_\alpha > \rho_\beta$ for every class $\beta \neq \alpha$ which has access to α . In particular, if α is not accessible from any other class, then α is distinguished. A real number λ is called a *distinguished eigenvalue* of M if there exists a non-zero vector $\mathbf{x} \geq \mathbf{0}$ such that $M\mathbf{x} = \lambda\mathbf{x}$. The following theorem extends the Perron-Frobenius Theorem to reducible matrices, see [23], [28] and [30] for the proof.

Theorem 2.13. *Let $M \in \mathcal{M}_{k \times k}(\mathbb{Z}_+)$.*

- (i) *A real number λ is a distinguished eigenvalue if and only if there exists a distinguished class α for M such that $\rho_\alpha = \lambda$.*
- (ii) *If α is a distinguished class of $G(M)$, then there exists a unique (up to scaling) non-negative eigenvector $\mathbf{v}_\alpha = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ corresponding to ρ_α with the property that $\mathbf{v}_i > 0$ if and only if the class α is accessible from the vertex i .*

We call \mathbf{v}_α the *distinguished eigenvector* of M corresponding to α .

Definition 2.14. *Let $M \in \mathcal{M}_{k \times k}(\mathbb{Z}_+)$. The core of M is $\text{core}(M) = \bigcap_{n \geq 1} M^n(\mathbb{R}_+^k)$.*

The next theorem can be found in [27].

Theorem 2.15. *Let M be a non-negative integer square matrix verifying (2.2). Then the core of M is the cone generated by the distinguished eigenvectors of M .*

2.6. Core of the substitution matrix. Let \mathcal{A} be a finite set of prototiles and let $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ be a substitution with expansion map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and substitution matrix $M_\omega = M \in \mathcal{M}_{\mathcal{A} \times \mathcal{A}}(\mathbb{Z}_+)$. (Here we abuse notation a little and label the rows and columns of M by the elements of the prototile set \mathcal{A} .) Suppose that $\alpha_1, \dots, \alpha_l$ are the equivalence classes (irreducible components) for the matrix M . For every $1 \leq i \leq l$ we denote by M_i the restriction of M to the class α_i . Let X_1, \dots, X_m be the minimal components of $X_{\mathcal{A}, \omega}$, and let $\mathcal{A}_i \subseteq \mathcal{A}$ be the minimal subset such that $X_i \subseteq X_{\mathcal{A}_i}$. As already mentioned, we may assume without loss of generality that $X_i = X_{\mathcal{A}_i, \omega}$ and $\omega|_{\mathcal{A}_i} : \mathcal{A}_i \rightarrow \mathcal{A}_i^+$ is primitive, for every $1 \leq i \leq m$. Observe that for every $1 \leq i \leq m$, the set of prototiles \mathcal{A}_i is equal to an equivalence class α_i of M . The restriction M_i of M to \mathcal{A}_i is the matrix associated to the substitution $\omega|_{\mathcal{A}_i}$. The equivalence classes of the matrix M can be ordered in such a way that M has the block upper-triangular form

$$M = \begin{pmatrix} M_1 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & M_2 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & M_m & * & \cdots & * \\ 0 & 0 & \cdots & 0 & M_{m+1} & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & M_l \end{pmatrix}$$

where $*$ stand for arbitrary non-negative integer matrices. Observe that for each $m+1 \leq i \leq l$ there must be at least one non-zero off-diagonal block, because those M_i correspond to non-minimal prototile subsets.

The number $\lambda_0 = |\det(\varphi)|$ is a left eigenvalue of M with $\mathbf{v}_0 = (\text{vol}(A))_{A \in \mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}}$ as an associated eigenvector. Indeed, the A -coordinate of $\mathbf{v}_0^T M$ is equal to $\text{vol}(\omega(A))$ and $\text{vol}(\omega(A)) = \lambda_0 \text{vol}(A)$, for every $A \in \mathcal{A}$ by (2.1). The next proposition shows that λ_0 is the unique distinguished eigenvalue of M_ω and characterizes the core of M_ω .

Proposition 2.16. *Let $M \in \mathcal{M}_{\mathcal{A} \times \mathcal{A}}(\mathbb{Z}_+)$ be the substitution matrix associated to the substitution $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ with expansion map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Assume that (2.2) holds. Let X_1, \dots, X_m be the minimal components of $X_{\mathcal{A}, \omega}$, and let \mathcal{A}_i be the corresponding prototile sets, so that $X_i = X_{\mathcal{A}_i}$. Then*

- (i) $\lambda_0 = |\det(\varphi)|$ is the unique distinguished eigenvalue of M .
- (ii) $\mathcal{A}_1, \dots, \mathcal{A}_m$ are all the different distinguished classes of M , and $\rho(M_i) = |\det(\varphi)|$ for every $1 \leq i \leq m$.
- (iii) Let $1 \leq i \leq m$. If \mathbf{v}_i is the distinguished eigenvector of M corresponding to the class \mathcal{A}_i , then $\mathbf{v}_i(p) > 0$ if and only if $p \in \mathcal{A}_i$.
- (iv) $\text{core}(M)$ is the cone generated by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$.

Proof. We know that M_i , for $i = 1, \dots, m$, is the substitution matrix for the primitive tile substitution $\omega|_{\mathcal{A}_i}$, hence $\rho(M_i)$ are distinguished eigenvalues of M . However, $\rho(M_i) = \rho(M_i^T) = |\det(\varphi)| = \lambda_0$ for $i = 1, \dots, m$, by the definition of tile substitution (2.1). On the other hand, the fact that M^T has a strictly positive eigenvector implies, by [11, Theorem 13.4.6] that $\rho(M_i) = \rho(M_i^T) < \rho(M^T) = \rho(M)$ for $i = m+1, \dots, l$. (This is also easy to see directly: every non-minimal class i “loses entropy” under the substitution, in view of $\omega(\mathcal{A}_i) \not\subseteq \mathcal{A}_i$.) Therefore, the classes $i = m+1, \dots, l$ are not distinguished by definition. Now all the claims of the proposition immediately follow from Theorem 2.13 and Theorem 2.15. \square

2.7. Invariant measures and transverse measures. Let (X, \mathbb{R}^d) be a tiling dynamical system (it need not be a substitution system, although for our purposes we can take $X = X_{\mathcal{A}, \omega}$). An *invariant measure* of (X, \mathbb{R}^d) is a measure $\mu : \mathcal{B}(X) \rightarrow \overline{\mathbb{R}}_+$ such that $\mu(U) = \mu(U - \mathbf{v})$, for every $U \in \mathcal{B}(X)$ and $\mathbf{v} \in \mathbb{R}^d$.

Recall that we consider every prototile $A \in \mathcal{A}$ centered at $\mathbf{0}$. The *center of the tile* $t \in \mathcal{T} \in X$ is the point $\mathbf{x}_t \in \mathbb{R}^d$ for which there exists $A \in \mathcal{A}$ such that $t = A + \mathbf{x}_t$. Let $\eta > 0$ be such that the interior of the support of A contains the closure of $B_\eta(\mathbf{0})$, for every $A \in \mathcal{A}$.

By the *transversal* of X we mean the set $\Gamma \subseteq X$ of all the tilings \mathcal{T} in X for which there exists a prototile $A \in \mathcal{A}$ such that $A \in \mathcal{T}$. In other words, if $L_\mathcal{T} \subseteq \mathbb{R}^d$ is the set of all the points $\mathbf{x} \in \mathbb{R}^d$ for which there exist $\mathcal{T} \in X$ and $A \in \mathcal{A}$ such that $A + \mathbf{x}$ is a tile of \mathcal{T} , then

$$\Gamma = \{\mathcal{T} \in X : \mathbf{0} \in L_\mathcal{T}\}.$$

We say that a patch P is X -admissible if there exists $\mathcal{T} \in X$ such that $P \subseteq \mathcal{T}$. We denote by Λ_X the collection of all the X -admissible patches P for which there exists a prototile $A \in \mathcal{A}$ such that $A \in P$. In other words, this is the collection of patches such that $\mathbf{0} \in \text{int}(\text{supp}(P))$ and $\mathbf{0}$ coincides with the center of some tile $t \in P$. Every X -admissible patch is a translate of some patch in Λ_X . For every $P \in \Lambda_X$ we define

$$C_P = \{\mathcal{T} \in X : P \subseteq \mathcal{T}\}.$$

The sets C_P are subsets of Γ .

Equipped with the induced topology, the space Γ is compact and totally disconnected, with a countable base of clopen sets (the collection of the sets C_P is a base of the topology). The collection of Borel sets $\mathcal{B}(\Gamma)$ of Γ is equal to $\mathcal{B}(X) \cap \Gamma$.

Remark 2.17. If $U \in \mathcal{B}(\Gamma)$ and $\Theta \subseteq \mathbb{R}^d$ is an open set, then $U + \Theta \in \mathcal{B}(X)$. To verify that, observe that this is true for the sets C_P , with $P \in \Lambda_X$. Verifying that the collection $\mathcal{M} = \{U \in \mathcal{B}(\Gamma) : U + \Theta \in \mathcal{B}(X)\}$ is a σ -algebra, we get the result.

A *transverse measure* on $\mathcal{B}(\Gamma)$ is a measure $\nu : \mathcal{B}(\Gamma) \rightarrow \overline{\mathbb{R}}_+$ such that $\nu(A) = \nu(A - \mathbf{v})$, for every $A \subseteq \mathcal{B}(\Gamma)$ and $\mathbf{v} \in \mathbb{R}^d$ for which $A - \mathbf{v} \subseteq \Gamma$.

Tiling spaces have been studied as *laminations*, or *translation surfaces*, see [3, 2]. Our definition agrees with the notion of transverse measure for laminations. There is a one-to-one correspondence between finite invariant measures and finite transverse measures (see [2, Section 5]), however, in all the existing literature it is assumed that the tiling system is

minimal. We need this correspondence in the non-minimal case, and also for σ -finite positive measures. Therefore, we include details about this in the Appendix (Section 7) for the reader's convenience. If μ is an invariant measure, we denote by μ^T the associated transverse measure.

3. FINITE INVARIANT MEASURES.

Theorem 3.1. *Let $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ be a self-affine tiling dynamical system. Then all finite invariant measures are supported on the minimal components.*

Scheme of the proof. Let μ be a finite invariant measure. We can assume that μ is ergodic. It is easy to see that the sets C_P , where P is an admissible patch, generate the Borel σ -algebra on Γ . Therefore, the Borel σ -algebra on $X_{\mathcal{A},\omega}$ is generated by the sets $C_P + \Theta$, with $\Theta \subseteq B_r(\mathbf{0})$, and their translates. The claim will follow once we show that $\mu(C_P + \Theta) = 0$ for every patch P which does not occur in $\bigcup_{i=1}^m X_i$, the union of minimal components. To this end, we will use the pointwise ergodic theorem and show that the frequency of such a patch P in any tiling $\mathcal{T} \in X_{\mathcal{A},\omega}$ equals zero. We have two cases to consider: (a) P contains a tile from $\mathcal{A}' := \mathcal{A} \setminus \bigcup_{i=1}^m \mathcal{A}_i$; (b) P contains tiles from two distinct minimal components. The following example shows why case (b) is needed.

Example 3.2. All the tiles have the unit square as its support and are distinguished only by the labels. Let $\mathcal{A} = \{\boxed{0}, \boxed{1}, \boxed{2}\}$; the substitution ω is given by

$$\boxed{0} \rightarrow \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}, \quad \boxed{1} \rightarrow \boxed{1}^{4 \times 4}, \quad \boxed{2} \rightarrow \boxed{2}^{4 \times 4}.$$

where $\boxed{a}^{k \times k}$ stands for the $k \times k$ square filled with identical prototiles labelled a . We have two minimal components, corresponding to the prototiles labelled 1 and 2. The tiling space $X_{\mathcal{A},\omega}$ contains tilings which have a half-plane filled by 1's and another half-plane filled by 2's. As we show, such tilings have zero measure for any finite invariant measure μ .

Now we turn to the details. We use the pointwise ergodic theorem for \mathbb{R}^d -actions, which was first proved by Wiener [32], with averaging over balls centered at the origin. For us, another averaging sequence is more convenient. Let

$$(3.1) \quad F_n := \varphi^n(B_1(\mathbf{0})).$$

It is well-known that the pointwise ergodic theorem holds with F_n used instead of the balls (see e.g. [29, 7]).

Theorem 3.3. *(Pointwise Ergodic Theorem for \mathbb{R}^d -actions) Let μ be an ergodic invariant probability measure for the system $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$. Then for any $f \in L^1(X_{\mathcal{A},\omega}, \mu)$,*

$$(3.2) \quad \frac{1}{\text{vol}(F_n)} \int_{F_n} f(\mathcal{T} - \mathbf{g}) d\mathbf{g} \rightarrow \int f(\mathcal{S}) d\mu(\mathcal{S}), \quad \text{as } n \rightarrow \infty,$$

for μ -a.e. $\mathcal{T} \in X_{\mathcal{A},\omega}$.

For a bounded set $F \subseteq \mathbb{R}^d$ and $r > 0$ let

$$F^{+r} := \{\mathbf{x} \in \mathbb{R}^d : \text{dist}(\mathbf{x}, F) \leq r\}.$$

Definition 3.4. A van Hove (Følner) sequence for \mathbb{R}^d is a sequence $\{\Theta_n\}_{n \geq 1}$ of bounded measurable subsets of \mathbb{R}^d satisfying

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{\text{vol}((\partial\Theta_n)^{+r})}{\text{vol}(\Theta_n)} = 0, \quad \text{for all } r > 0.$$

It is easy to see that our sequence $\{F_n\}_{n \geq 1}$ defined in (3.1), is van Hove. Moreover, $\{\varphi^n(\text{supp}(t))\}_{n \geq 1}$ is van Hove for any prototile $t \in \mathcal{A}_i$, $i \leq m$. Indeed, the latter follows from the fact that $\text{vol}(\partial(\text{supp}(t))) = 0$ for a prototile in a minimal component, which was proved for primitive tile substitutions in [18, Prop.1.1].

Given a set $F \subseteq \mathbb{R}^d$, patch P , and a tiling \mathcal{T} , denote by $N(F, P, \mathcal{T})$ the number of patches in \mathcal{T} equivalent to P such that $\text{supp}(P) \subseteq F$. Note that for any set F ,

$$(3.4) \quad N(F, P, \mathcal{T}) \leq \delta^{-1} \text{vol}(F),$$

where δ is the volume of a ball contained in the interior of every prototile. For $\mathcal{T} \in X_{\mathcal{A},\omega}$ and a patch P , the frequency of P in \mathcal{T} with respect to F_n (which is our default averaging sequence) is defined by

$$\text{freq}_{\mathcal{T}}(P) = \lim_{n \rightarrow \infty} \frac{N(F_n, P, \mathcal{T})}{\text{vol}(F_n)},$$

whenever the limit exists.

For f the indicator function of the set $C_P + \Theta$ and $\beta = \text{diam}(P) + \text{diam}(\Theta)$ we obtain

$$\begin{aligned} 0 \leq \frac{1}{\text{vol}(F_n)} \int_{F_n} f(\mathcal{T} - \mathbf{t}) d\mathbf{t} &\leq \frac{N((F_n)^{+\beta}, P, \mathcal{T})}{\text{vol}(F_n)} \cdot \text{vol}(\Theta) \\ &\leq \left(\frac{N(F_n, P, \mathcal{T})}{\text{vol}(F_n)} + \delta^{-1} \frac{\text{vol}((\partial F_n)^{+\beta})}{\text{vol}(F_n)} \right) \cdot \text{vol}(\Theta) \end{aligned}$$

hence $\text{freq}_{\mathcal{T}}(P) = 0$ for μ -a.e. \mathcal{T} will imply $\mu(C_P + \Theta) = 0$ by (3.2), in view of $\{F_n\}$ being van Hove.

Lemma 3.5. Let M be the substitution matrix for a tile substitution with expansion map φ . Then for every $A \in \mathcal{A}'$ and $B \in \mathcal{A}$,

$$\lim_{n \rightarrow \infty} \frac{M^n(A, B)}{|\det(\varphi)|^n} = 0.$$

Proof. This follows from the structure of the matrix M and Proposition 2.16, using that all classes in \mathcal{A}' are non-distinguished. A direct reference is [23, Theorem 9.4]. \square

Lemma 3.6. Let A be a tile in \mathcal{A}' and $\mathcal{T} \in X_{\mathcal{A},\omega}$. Then $\text{freq}_{\mathcal{T}}(A) = 0$.

Proof. Fix $n \in \mathbb{N}$. Recall that $\omega : X_{\mathcal{A},\omega} \rightarrow X_{\mathcal{A},\omega}$ is onto, hence there exists $\mathcal{T}' \in X_{\mathcal{A},\omega}$ such that $\omega^n(\mathcal{T}') = \mathcal{T}$. Observe that

$$F_n = \varphi^n(B_1(\mathbf{0})) \subseteq \text{supp}(\omega^n([B_1(\mathbf{0})]^{\mathcal{T}'})).$$

Let $[B_1(\mathbf{0})]^{\mathcal{T}'} = \{t_1 + \mathbf{x}_1, \dots, t_k + \mathbf{x}_k\}$ where t_i are prototiles (not necessarily distinct) and $\mathbf{x}_i \in \mathbb{R}^d$. Note that $k \leq K$ where K is a uniform constant by the FPC property. We have

$$\frac{N(F_n, A, \mathcal{T})}{\text{vol}(F_n)} \leq \frac{\sum_{i=1}^k N(\omega^n(t_i + \mathbf{x}_i), A, \mathcal{T})}{\text{vol}(F_n)} = \frac{\sum_{i=1}^k M^n(A, t_i)}{|\det(\varphi)|^n \text{vol}(B_1(\mathbf{0}))},$$

and the claim follows from Lemma 3.5. \square

Lemma 3.7. *Let P be an $X_{\mathcal{A}, \omega}$ -admissible patch such that all its prototiles belong to $\mathcal{A} \setminus \mathcal{A}'$ but P is not admissible for any of the minimal components X_i . Then $\text{freq}_{\mathcal{T}}(P) = 0$ for any $\mathcal{T} \in X_{\mathcal{A}, \omega}$.*

Proof. Let $n = s + \ell$ and find $\mathcal{T}' \in X_{\mathcal{A}, \omega}$ such that $\omega^n(\mathcal{T}') = \mathcal{T}$. Consider

$$C := \omega^s([B_1(\mathbf{0})]^{\mathcal{T}'}) = \{t_{1,s} + \mathbf{y}_{1,s}, \dots, t_{k_s,s} + \mathbf{y}_{k_s,s}\},$$

where $t_{i,s}$ are prototiles (not necessarily distinct) and $\mathbf{y}_{i,s} \in \mathbb{R}^d$. Then $F_n \subseteq \text{supp}(\omega^\ell(C))$ and we have

$$N(F_n, P, \mathcal{T}) \leq N(\omega^\ell(C), P, \mathcal{T}) = N\left(\bigcup_{i=1}^{k_s} \omega^\ell(t_{i,s} + \mathbf{y}_{i,s}), P, \mathcal{T}\right).$$

Observe that if P has nonempty intersection with $\omega^\ell(t_{i,s} + \mathbf{y}_{i,s})$, then either $t_{i,s} \in \mathcal{A}'$ or $\text{supp}(P)$ intersects the boundary $\partial(\text{supp}(\omega^\ell(t_{i,s} + \mathbf{y}_{i,s})))$ for $t_{i,s} \notin \mathcal{A}'$. Thus, in view of (3.4),

$$(3.5) \quad N(F_n, P, \mathcal{T}) \leq \delta^{-1} \sum_{i \leq k_s: t_{i,s} \in \mathcal{A}'} \text{vol}(\text{supp}(\omega^\ell(t_{i,s}))) + \delta^{-1} \sum_{i \leq k_s: t_{i,s} \notin \mathcal{A}'} \text{vol}((\partial(\text{supp}(\omega^\ell(t_{i,s}))))^{+\alpha}))$$

where $\alpha = \text{diam}(P)$. Fix $\varepsilon > 0$. By Lemma 3.6, there exists s_0 such that for $s \geq s_0$ we have

$$\#\{i \leq k_s : t_{i,s} \in \mathcal{A}'\} \leq \varepsilon |\det \varphi|^s.$$

Denoting by V_{\max} the maximal volume of a prototile we obtain

$$(3.6) \quad \sum_{i: t_{i,s} \in \mathcal{A}'} \text{vol}(\text{supp}(\omega^\ell(t_{i,s}))) \leq \varepsilon |\det \varphi|^s \max_{i,s} \text{vol}(\text{supp}(\omega^\ell(t_{i,s}))) \leq \varepsilon |\det \varphi|^{s+\ell} V_{\max}.$$

On the other hand, $\{\text{supp}(\omega^\ell(t))\}_{\ell \geq 1}$ is a van Hove sequence for any prototile $t \notin \mathcal{A}'$, hence there exists ℓ_0 such that for any $t \notin \mathcal{A}'$, for $\ell \geq \ell_0$,

$$\text{vol}((\partial(\text{supp}(\omega^\ell(t))))^{+\alpha}) \leq \varepsilon \text{vol}(\text{supp}(\omega^\ell(t))).$$

Combining this with (3.5), (3.6), and using that

$$\sum_{i=1}^{k_s} \text{vol}(\omega^\ell(t_{i,s})) = \text{vol}(\omega^{s+\ell}([B_1(\mathbf{0})]^{\mathcal{T}'}) \leq |\det \varphi|^{s+\ell} K V_{\max}$$

yields

$$N(F_{s+\ell}, P, \mathcal{T}) \leq \varepsilon \delta^{-1} |\det \varphi|^{s+\ell} (1 + K) V_{\max} \quad \text{for } s \geq s_0, \ell \geq \ell_0,$$

and the claim follows. \square

Now Theorem 3.1 is proved by the scheme given after its statement. We also obtain the following

Theorem 3.8. *There is an affine bijection between the set of finite invariant measures of $X_{\mathcal{A},\omega}$ and $\text{core}(M)$. The finite ergodic measures of $X_{\mathcal{A},\omega}$ are in one-to-one correspondence with the distinguished eigenvectors of M .*

Proof. Let μ be a finite ergodic measure. Theorem 3.1 implies that μ is supported on a minimal component X_i . Since X_i is a minimal substitution system, [24, Theorem 3.1]), [24, theorem 3.3], [24, Corollary 3.5] and [14, 8.2.11] imply that μ is determined by the Perron eigenvalue of the matrix M_i , where M_i is the restriction of M_ω to the minimal component X_i . Indeed, $(\mu^T(C_A))_{A \in \mathcal{A}_i}$ is a Perron eigenvector of M_i . Since $\mu^T(C_D) = 0$ for every $D \in \mathcal{A}'$, Theorem 2.13 implies that $(\mu^T(C_A))_{A \in \mathcal{A}}$ is a distinguished eigenvector of M associated to the class \mathcal{A}_i . Since $\text{core}(M)$ is the cone generated by its distinguished eigenvectors, we get that $(\mu^T(C_A))_{A \in \mathcal{A}}$ is in $\text{core}(M)$. It is straightforward to show that the function $\mu \mapsto (\mu^T(C_A))_{A \in \mathcal{A}}$, defined on the set of finite invariant measures of $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ to $\text{core}(M)$, is affine and bijective. \square

4. RECOGNIZABILITY

A substitution ω is *non-periodic* if the dynamical system $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ has no periodic points, that is, if $\mathcal{T} \in X_{\mathcal{A},\omega}$ and $\mathcal{T} + \mathbf{v} = \mathcal{T}$ for $\mathbf{v} \in \mathbb{R}^d$, then $\mathbf{v} = \mathbf{0}$. In this section we show the following theorem:

Theorem 4.1. *Let $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ be an admissible tiling substitution. The function $\omega : X_{\mathcal{A},\omega} \rightarrow X_{\mathcal{A},\omega}$ is one-to-one if and only if ω is non-periodic.*

It is straightforward to show that a periodic substitution is not one-to-one. Indeed, if $\mathcal{T} \in X_{\mathcal{A},\omega}$ is such that $\mathcal{T} = \mathcal{T} + \mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, then Proposition 2.8 implies that for every $i \geq 1$ there exists $\mathcal{T}_i \in X_{\mathcal{A},\omega}$ such that $\omega^i(\mathcal{T}_i) = \omega^i(\mathcal{T}_i + \varphi^{-i}(\mathbf{v})) = \mathcal{T}$. Observe that for $i \geq 1$ sufficiently large, $\mathcal{T}_i \neq \mathcal{T}_i + \varphi^{-i}(\mathbf{v})$ since $\varphi^{-i}(\mathbf{v})$ is close to zero. This proves that ω is not one-to-one.

Theorem 4.1 was already proved for primitive substitutions in [25], so here we focus on the non-primitive case.

We also obtain “partial recognizability” for a class of substitutions with periodic points which we now define. We can assume, without loss of generality, that (2.2) holds, and X_1, \dots, X_m are the minimal components of $X_{\mathcal{A},\omega}$. We say that X_j is *periodic* if there exists $\mathcal{T} \in X_j$ and $\mathbf{v} \neq \mathbf{0}$ such that $\mathcal{T} + \mathbf{v} = \mathcal{T}$. Then \mathbf{v} is a translational period for all tilings in X_j . Denote by \mathcal{A}_{per} the set of prototiles which occur in periodic minimal components and $\mathcal{A}_{\text{nonp}} := \mathcal{A} \setminus \mathcal{A}_{\text{per}}$. For any legal patch P let $P|_{\text{nonp}}$ be the subpatch of all $\mathcal{A}_{\text{nonp}}$ tiles in P .

Definition 4.2. *We say that a substitution ω satisfies the non-periodic border condition if*

$$(4.1) \quad \forall A \in \mathcal{A}_{\text{nonp}}, \quad \partial(\text{supp}(\omega(A))) \subseteq \text{supp}(\omega(A)|_{\text{nonp}}).$$

Definition 4.3. *A tile substitution ω is said to be partially recognizable if for every $\mathcal{T} \in X_{\mathcal{A},\omega}$ which contains a tile of $\mathcal{A}_{\text{nonp}}$ type there is a unique tiling $\mathcal{T}' \in X_{\mathcal{A},\omega}$ such that $\omega(\mathcal{T}') = \mathcal{T}$.*

Theorem 4.4. *An admissible tile substitution ω satisfying the non-periodic border condition is partially recognizable.*

Corollary 4.5. *If ω has non-periodic border and $\mathcal{T} \in X_{\mathcal{A}, \omega}$ contains a tile in $\mathcal{A}_{\text{nonp}}$, then \mathcal{T} is non-periodic.*

The non-periodic border condition is not necessary for the claim of Theorem 4.4 to hold (see Example 6.4 below), but the following example shows that some assumption on the substitution ω is needed. It is plausible that, without any additional assumptions, a *non-periodic* tiling \mathcal{T} has a unique preimage under ω ; however, this remains an open question.

Example 4.6. Let $\mathcal{A} = \{\boxed{0}, \boxed{1}\}$;

$$\boxed{0} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \quad \boxed{1} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

Note that all tilings in $X_{\mathcal{A}, \omega}$ are periodic under the vertical shift, but according to our definition, only the prototile labelled 0 (which is in the minimal component) is periodic. Thus, the conclusion of Corollary 4.5 is violated (of course, the non-periodic border condition does not hold).

Now we start preparation for the proofs. Clearly, a non-periodic substitution has non-periodic border, so Theorem 4.4 implies Theorem 4.1. Our argument is based on the method of [15] where a new proof of recognizability for primitive tile substitutions was given (it applied to a more general class of tilings, not just translationally FPC). We should note that a direct proof of Theorem 4.1 is simpler, and we will indicate which parts can be skipped if one is only interested in non-periodic substitutions.

Recall that $\eta > 0$ is such that the support of every prototile in \mathcal{A} contains the closed ball $\overline{B}_\eta(\mathbf{0})$ in its interior. Let $\gamma = \max_{A \in \mathcal{A}}\{\text{diam}(A)\}$, and $1 < \lambda_1 \leq \lambda_2 < \infty$ are positive numbers such that

$$(4.2) \quad \lambda_1 \|\mathbf{x}\| \leq \|\varphi(\mathbf{x})\| \leq \lambda_2 \|\mathbf{x}\|, \text{ for every } \mathbf{x} \in \mathbb{R}^d.$$

Since φ is expansive, we can find a norm in \mathbb{R}^d with this property, and balls in this section will always be considered with respect to this norm. Then for every $n \geq 1$ and $\mathbf{y} \in \mathbb{R}^d$,

$$(4.3) \quad B_{\lambda_1^n r}(\varphi^n(\mathbf{y})) \subseteq \varphi^n(B_r(\mathbf{y})) \subseteq B_{\lambda_2^n r}(\varphi^n(\mathbf{y}))$$

The next definition was introduced in [15].

Let W be an $X_{\mathcal{A}, \omega}$ -admissible patch. For every $n \geq 0$, let $\mathcal{P}_n(W)$ be the set of $X_{\mathcal{A}, \omega}$ -admissible patches P satisfying:

- (1) $\omega^n(W) \subseteq \omega^n(P)$;
- (2) $\omega^n(W)$ is not included in $\omega^n(P')$, for any proper subpatch P' of P .

Actually, [15] used only $\mathcal{P}_n(t)$ for a single tile t ; this would be sufficient if we were to restrict ourselves to non-periodic tilings.

Lemma 4.7. *Let W be an $X_{\mathcal{A},\omega}$ -admissible patch.*

- (i) *Let $n \geq 0$. For every $P \in \mathcal{P}_n(W)$, we have $\text{supp}(P) \subseteq B_{2\gamma}(\text{supp}(W))$.*
- (ii) *$\{W\} = \mathcal{P}_0(W) \subseteq \mathcal{P}_1(W) \subseteq \mathcal{P}_2(W) \subseteq \dots$*

Proof. (i) Let $n \geq 0$ and $P \in \mathcal{P}_n(W)$. Since $\omega^n(W) \subseteq \omega^n(P)$, we have that $\text{supp}(W) \subseteq \text{supp}(P)$. Let P' be the set of all tiles in P whose supports intersect $\text{supp}(W)$. Then $P' \subseteq P$, and we have $\omega^n(W) \subseteq \omega^n(P') \subseteq \omega^n(P)$. By the definition of $\mathcal{P}_n(W)$ we must have $P' = P$, and the desired property follows, since γ is the maximal diameter of a $X_{\mathcal{A},\omega}$ -tile.

(ii) It is clear that $\{W\} = \mathcal{P}_0(W)$. Let $n \geq 0$ and $P \in \mathcal{P}_n(W)$. The definition of the set $\mathcal{P}_n(W)$ implies that $\omega^{n+1}(W) \subseteq \omega^{n+1}(P)$. If P' is a subpatch of P for which $\omega^{n+1}(W) \subseteq \omega^{n+1}(P')$, then the support of $\omega^n(W)$ is included in the support of $\omega^n(P')$. This implies that $\omega^n(W) \subseteq \omega^n(P')$, and from definition of $\mathcal{P}_n(W)$, we get $P' = P$. This shows that $\mathcal{P}_n(W) \subseteq \mathcal{P}_{n+1}(W)$. \square

Let $\mathcal{P}(W) = \bigcup_{n \geq 0} \mathcal{P}_n(W)$. The FPC assumption and part (i) of Lemma 4.7 imply that $\mathcal{P}(W)$ is finite up to translation. In the non-periodic case one can show that $\mathcal{P}(t)$ is finite, for any tile t . In the non-periodic border case, this is no longer true, and we have to work with “first coronas” or “collared tiles” containing at least one non-periodic tile. More precisely, consider all legal patches of the form $[\text{supp}(t)]^{\mathcal{T}}$, $t \in \mathcal{T}$, for some $\mathcal{T} \in X_{\mathcal{A},\omega}$; there are finitely many of them, up to translation. We choose a representative for each of the translation-equivalent classes, and denote their collection by \mathcal{F} . Denote by $\mathcal{F}_{\text{nonp}}$ the set of patches in \mathcal{F} which contain a tile of type $\mathcal{A}_{\text{nonp}}$. Now we can state the key proposition needed in the proof of Theorem 4.4.

Proposition 4.8. *There exists $M \in \mathbb{N}$ such that for any $\mathcal{T} \in X_{\mathcal{A},\omega}$, $n \geq 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, if $P = \omega^n(Z)$, with $Z - \mathbf{y} \in \mathcal{F}_{\text{nonp}}$, such that*

$$P \subseteq \mathcal{T}, \quad P + \mathbf{x} \subseteq \mathcal{T},$$

then

$$\mathbf{x} \in \varphi^{n-M}(B_{\eta}(\mathbf{0})) \text{ implies } \mathbf{x} = \mathbf{0}.$$

The proof will be based on several lemmas.

Lemma 4.9. *There exists $R_0 > 0$ such that for every $\mathcal{T} \in X_{\mathcal{A},\omega}$ and $\mathbf{x} \in \mathbb{R}^d$, the ball $B_{R_0}(\mathbf{x})$ contains an X_i -admissible \mathcal{T} -patch, for some $1 \leq i \leq m$.*

Proof. Suppose that for every $R > 0$ there exist $\mathcal{T}_R \in X_{\mathcal{A},\omega}$ and $\mathbf{x}_R \in \mathbb{R}^d$, such that the ball $B_R(\mathbf{x}_R)$ does not contain patches belonging to any minimal component. Compactness of $X_{\mathcal{A},\omega}$ implies there exists a sequence $R_n \uparrow \infty$ such that $(\mathcal{T}_{R_n} - \mathbf{x}_{R_n})_{n \geq 0}$ converges to some tiling $\mathcal{T} \in X_{\mathcal{A},\omega}$. It follows that \mathcal{T} does not contain patches from any minimal component, which is not possible because $\text{Clos}\{\mathcal{T} - \mathbf{g} : \mathbf{g} \in \mathbb{R}^d\}$ must contain at least one minimal component of $X_{\mathcal{A},\omega}$. \square

The strategy of the proof of Proposition 4.8 is to find a large sub-patch of $\omega^n(t) \subseteq \omega^n(Z)$, with $t - \mathbf{x}_t \in \mathcal{F}_{\text{nonp}}$, which belongs to a minimal component X_i . This component may be

non-periodic or periodic. The former case is treated with the following two lemmas. The first one is a special case of [25, Lemma 3.2].

Lemma 4.10. ([25, Lemma 3.2]) *Let $\omega : \mathcal{A} \rightarrow \mathcal{A}^+$ be a primitive non-periodic substitution, and let $\eta > 0$ be such that the support of every prototile in \mathcal{A} contains the ball $B_\eta(\mathbf{0})$. Then there exists $N \in \mathbb{N}$ such that, for any $\mathcal{T} \in X_{\mathcal{A}, \omega}$, $l > 0$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, if*

$$P \subseteq \mathcal{T}, \quad P + \mathbf{x} \in \mathcal{T}, \quad \varphi^l(B_\eta(\mathbf{0})) + \mathbf{y} \subseteq \text{supp}(P),$$

then

$$\mathbf{x} \in \varphi^{l-N}(B_\eta(\mathbf{0})) \text{ implies } \mathbf{x} = \mathbf{0}.$$

Lemma 4.11. *There exists $N \in \mathbb{N}$ such that for any $\mathcal{T} \in X_{\mathcal{A}, \omega}$, $n \geq 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, if P is an X_i -admissible patch, where X_i is a non-periodic minimal component, such that*

$$P \subseteq \mathcal{T}, \quad P + \mathbf{x} \subseteq \mathcal{T}, \quad \varphi^n(B_\eta(\mathbf{0})) + \mathbf{y} \subseteq \text{supp}(P),$$

then

$$\mathbf{x} \in \varphi^{n-N}(B_\eta(\mathbf{0})) \text{ implies } \mathbf{x} = \mathbf{0}.$$

Proof. Let $k \in \mathbb{N}$ be such that $2B_\eta(\mathbf{0}) = B_\eta(\mathbf{0}) + B_\eta(\mathbf{0}) \subseteq \varphi^k(B_\eta(\mathbf{0}))$. We can take $k = \lceil \log 2 / \log \lambda_1 \rceil$ by (4.2). By Lemma 4.10, for $j \in \{1, \dots, m\}$, there exists $N_j \in \mathbb{N}$ such that if Q is an X_j -admissible patch satisfying

$$Q \subseteq \mathcal{T}', \quad Q + \mathbf{w} \subseteq \mathcal{T}', \quad \varphi^n(B_\eta(\mathbf{0})) + \mathbf{v} \subseteq \text{supp}(Q),$$

for some $\mathbf{w} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\mathbf{v} \in \mathbb{R}^d$ and $\mathcal{T}' \in X_j$, then

$$\mathbf{w} \notin \varphi^{n-N_j}(B_\eta(\mathbf{0})).$$

We claim that the desired statement holds for $N = k + \max\{N_j : 1 \leq j \leq m\}$. Let $\mathcal{T} \in X_{\mathcal{A}, \omega}$, $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and P be an X_i -admissible patch such that

$$P \subseteq \mathcal{T}, \quad P + \mathbf{x} \subseteq \mathcal{T}, \quad \varphi^n(B_\eta(\mathbf{0})) + \mathbf{y} \subseteq \text{supp}(P),$$

for some $n \geq 0$ and $\mathbf{y} \in \mathbb{R}^d$. Further, suppose that $\mathbf{x} \in \varphi^{n-N}(B_\eta(\mathbf{0})) \setminus \{\mathbf{0}\}$. Clearly, $n > N$, since every tile contains a ball of radius r , so shifting a tile by a vector in $B_\eta(\mathbf{0})$ will result in a tile intersecting the original one, making $P, P + \mathbf{x} \subseteq \mathcal{T}$ impossible. Since P is X_i -admissible, there exists $\mathcal{T}' \in X_i$ such that $P \subseteq \mathcal{T}'$. Consider

$$P' := [\varphi^{n-k}(B_\eta(\mathbf{0})) + \mathbf{y}]^{\mathcal{T}} \subseteq \mathcal{T}'.$$

We want to apply Lemma 4.11 to P' and \mathcal{T}' . The only thing we need to check is that $P' + \mathbf{x} \subseteq \mathcal{T}'$. This will follow if we show that $P' + \mathbf{x} \subseteq P$, and to verify the latter it suffices to check that

$$\varphi^{n-k}(B_\eta(\mathbf{0})) + \mathbf{y} + \mathbf{x} \subseteq \varphi^n(B_\eta(\mathbf{0})) + \mathbf{y}.$$

However,

$$\varphi^{n-k}(B_\eta(\mathbf{0})) + \mathbf{x} \subseteq \varphi^{n-k}(B_\eta(\mathbf{0})) + \varphi^{n-N}(B_\eta(\mathbf{0})) \subseteq \varphi^{n-k}(B_\eta(\mathbf{0})) + \varphi^{n-k}(B_\eta(\mathbf{0})) \subseteq \varphi^n(B_\eta(\mathbf{0}))$$

by the definition of k . The proof is complete. \square

Proof of Proposition 4.8. Suppose that \mathcal{T} , P , and \mathbf{x} are as in the statement of the proposition, with some $M \geq 1$, which will be determined below. Suppose that $\mathbf{x} \neq \mathbf{0}$. Let t be a tile of the patch Z of type $\mathcal{A}_{\text{nonp}}$. Its support contains the ball $B_\eta(\mathbf{y})$ for some $\mathbf{y} \in \mathbb{R}^d$. Then

$$\text{supp}(P) = \text{supp}(\omega^n(Z)) \supseteq \text{supp}(\omega^n(t)) \supseteq \varphi^n(B_\eta(\mathbf{y})).$$

Let $n_0 \geq 0$ be the smallest integer satisfying $\eta \lambda_1^{n_0} \geq R_0$, that is, $n_0 = \lceil \log_{\lambda_1} \left(\frac{R_0}{\eta} \right) \rceil$. Here R_0 is the constant from Lemma 4.9. Let \mathcal{S} be any tiling in $X_{\mathcal{A}, \omega}$ satisfying $\omega^{n-n_0}(\mathcal{S}) = \mathcal{T}$. Then $\varphi^{n_0}(B_\eta(\mathbf{y}))$ contains a ball of radius R_0 , hence an X_i -admissible tile $t' \in \mathcal{S}$, for some minimal component X_i , $1 \leq i \leq m$, by Lemma 4.9. Therefore, $\varphi^n(B_\eta(\mathbf{y}))$, and hence the patch $\omega^n(t) \subseteq P$, contains an X_i -admissible patch $P' = \omega^{n-n_0}(t') \subseteq \mathcal{T}$. Now there are two cases. If X_i is non-periodic, then we apply Lemma 4.10 to conclude that $\mathbf{x} = \mathbf{0}$, provided $M \geq N + n_0$ (where N is from Lemma 4.11). **This concludes the proof in the case when the substitution ω is non-periodic.**

It remains to treat the case when X_i is periodic. The idea is the following: since $P, P+\mathbf{x} \subseteq \mathcal{T}$, we have that $P' + \mathbf{x} \subseteq P$ as long as $\text{supp}(P' + \mathbf{x}) \subseteq \text{supp}(P)$. Then $P' + \mathbf{x}$ is also X_i -admissible. We can continue in this manner as long as the translates of P' by a multiple of \mathbf{x} remain in $\text{supp}(P)$, and this works for individual tiles as well, not necessarily for the entire P' . If \mathbf{x} is small relative to the size of P' , we will obtain an entire “tube” from P' to the border of $\omega^n(t)$ which is X_i -admissible. But this will lead to a contradiction with the non-periodic border assumption. Now for the details. A slight complication arises because of the possibility that the interior of a tile is disconnected, so we actually take the “connected component” of P' .

Let us continue with the proof of the proposition. We can assume that $M \geq n_0$. Clearly, the assumptions imply $n \geq M$ (since $\mathbf{x} \in \varphi^{n-M}(B_\eta(\mathbf{0}))$ is a non-zero translation between two tiles in \mathcal{T} , and every prototile contains $B_\eta(\mathbf{0})$ in the interior of its support). Recall that $P' = \varphi^{n-n_0}(t')$ is X_i -admissible, where X_i is a periodic minimal component, $P' \subseteq \omega^n(t) \subseteq P$, and t is a tile of type $\mathcal{A}_{\text{nonp}}$. It follows by induction from (4.1) that

$$(4.4) \quad \partial(\text{supp}(\omega^n(t))) \subseteq \text{supp}(\omega^n(t)|_{\text{nonp}}).$$

The tile t' contains a ball $B_\eta(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{R}^d$, hence $\varphi^{n-n_0}(B_\eta(\mathbf{z})) \subseteq \text{supp}(P')$. Consider

$$V := \text{the component of } \text{int}(\text{supp}(\omega^{n-n_0}(t'))) \text{ containing } \varphi^{n-n_0}(B_\eta(\mathbf{z})).$$

Clearly $[V]^\mathcal{T} \subseteq P'$, so all its tiles are of type $\mathcal{A}_{\text{nonp}}$. Note that $V \cap (V + \mathbf{x}) \neq \emptyset$ because

$$\varphi^{n-n_0}(B_\eta(\mathbf{z})) \cap (\varphi^{n-n_0}(B_\eta(\mathbf{z})) + \mathbf{x}) \neq \emptyset \text{ for } \mathbf{x} \in \varphi^{n-M}(B_\eta(\mathbf{0})) \subseteq \varphi^{n-n_0}(B_\eta(\mathbf{0})).$$

Let $k \geq 0$ be the largest integer such that $V + k\mathbf{x} \subseteq \text{supp}(\omega^n(t))$. Then $[V]^\mathcal{T} + (k+1)\mathbf{x} \subseteq \mathcal{T}$, because $\omega^n(t) \subseteq P$ and $P + \mathbf{x} \subseteq \mathcal{T}$. Moreover,

$$V + (k+1)\mathbf{x} \not\subseteq \text{supp}(\omega^n(t)) \text{ and } (V + k\mathbf{x}) \cap (V + (k+1)\mathbf{x}) \neq \emptyset.$$

It follows that $V + (k+1)\mathbf{x}$ contains a point \mathbf{z}_1 in the interior of a tile in $\mathcal{T} \setminus \omega^n(t)$ and $V + k\mathbf{x}$ contains a point \mathbf{z}_2 in the interior of a tile in $\omega^n(t)$. The set

$$W := (V + k\mathbf{x}) \cup (V + (k+1)\mathbf{x})$$

is open and connected, being a union of two open connected sets with non-empty intersection. Thus it is arcwise connected. An arc in W with the endpoints \mathbf{z}_1 and \mathbf{z}_2 must intersect the boundary of $\text{supp}(\omega^n(t))$. The point of intersection belongs to a tile, say, t'' , of type $\mathcal{A}_{\text{nonp}}$ by (4.4). This tile is in $[W]^\mathcal{T}$, but the types of all tiles in $[W]^\mathcal{T}$ are those of the tiles in $[V]^\mathcal{T}$, hence they are X_i -admissible. Thus, t'' is of type \mathcal{A}_{per} , which is a contradiction. \square

Lemma 4.12. *There exists $N \in \mathbb{N}$ such that for any patch Z and $\mathbf{y} \in \mathbb{R}^d$ such that $Z - \mathbf{y} \in \mathcal{F}_{\text{nonp}}$, we have $\mathcal{P}_{n+1}(Z) = \mathcal{P}_n(Z)$ for all $n \geq N$.*

Proof. Let $n \geq 0$ and $P \in \mathcal{P}_n(Z)$. Suppose that $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ is such that $P + \mathbf{x} \in \mathcal{P}_n(Z)$. We have

$$\omega^n(Z) \subseteq \omega^n(P) \quad \text{and} \quad \omega^n(Z) - \varphi^n(\mathbf{x}) \subseteq \omega^n(P).$$

We conclude from Lemma 4.11 that $\varphi^n(\mathbf{x}) \notin \varphi^{n-M}(B_\eta(\mathbf{0}))$, hence $\mathbf{x} \notin \varphi^{-M}(B_\eta(\mathbf{0}))$, which implies

$$\|\mathbf{x}\| \geq \eta \lambda_2^{-M}.$$

From part (i) of Lemma 4.7 it follows that the supports of translated copies of P which belong to $\mathcal{P}_n(Z)$, are contained in $B_{2\gamma}(\text{supp}(Z))$. Recall that $Z - \mathbf{y} \in \mathcal{F}_{\text{nonp}}$ is a “collared tile” hence $\text{diam}(Z) \leq 3\gamma$. Thus, there are at most

$$\frac{\text{vol}(B_{5\gamma + \eta \lambda_2^{-M}/2}(\mathbf{0}))}{\text{vol}(B_{\eta \lambda_2^{-M}/2}(\mathbf{0}))} =: N_1$$

copies of the patch P in $\mathcal{P}_n(Z)$. The FPC ensures that there are finitely many patches in $\mathcal{F}_{\text{nonp}}$, up to translation. Also by FPC, there are at most C patches, up to translation, whose support is contained in $B_{2\gamma}(\text{supp}(Z))$ for some $Z \in \mathcal{F}_{\text{nonp}}$. From this we deduce that $\mathcal{P}_n(Z)$ has at most CN_1 elements. Since this is valid for every $n \geq 0$, from part (ii) of Lemma 4.7 it follows that $|\mathcal{P}(Z)| \leq CN_1$. \square

We continue with the scheme of [15].

Lemma 4.13. *Suppose $\mathcal{P}_n(W) = \mathcal{P}_{n+1}(W)$, where W is a legal patch. If $\mathcal{S} \in X_{\mathcal{A}, \omega}$ is such that $\omega^{n+1}(W) \subseteq \omega(\mathcal{S})$, then $\omega^n(W) \subseteq \mathcal{S}$.*

Proof. Let $\mathcal{S}' \in X_{\mathcal{A}, \omega}$ be such that $\omega^n(\mathcal{S}') = \mathcal{S}$. Then $\omega^{n+1}(W) \subseteq \omega(\mathcal{S}) = \omega^{n+1}(\mathcal{S}')$, hence there exists $P \in \mathcal{P}_{n+1}(W)$ such that $P \subseteq \mathcal{S}'$. Since $P \in \mathcal{P}_n(W)$ we have $\omega^n(W) \subseteq \omega^n(P) \subseteq \omega^n(\mathcal{S}') = \mathcal{S}$. \square

Proof of Theorem 4.4. Let $\mathcal{T}_1 \in X_{\mathcal{A}, \omega}$ be such that $\omega(\mathcal{T}_1) = \mathcal{T}$, and further, suppose $\mathcal{T}_n \in X_{\mathcal{A}, \omega}$ are such that $\omega(\mathcal{T}_n) = \mathcal{T}_{n-1}$ for $n \geq 2$. Let $t_n \in \mathcal{T}_n$ be such that $\text{supp}(t_n) \ni \mathbf{0}$, and let $Z_n = [\text{supp}(t_n)]^{\mathcal{T}_n}$. Then $\omega^n(Z_{n+1}) \subseteq \mathcal{T}_1$ and

$$\bigcup_{n \geq 1} \text{supp}(\omega^n(Z_{n+1})) = \mathbb{R}^d,$$

hence Z_k are of type $\mathcal{F}_{\text{nonp}}$ for k sufficiently large (otherwise, \mathcal{T}_1 and $\mathcal{T} = \omega(\mathcal{T}_1)$ contain only tiles from \mathcal{A}_{per} contradicting our assumption), that is, there exists k_0 such that $Z_k - \mathbf{z}_k \in \mathcal{F}_{\text{nonp}}$ for $k \geq k_0$. We want to show that \mathcal{T}_1 , with $\omega(\mathcal{T}_1) = \mathcal{T}$, is uniquely determined. To this end,

consider any \mathcal{T}' , with $\omega(\mathcal{T}') = \mathcal{T}$. We have for $n \geq \max\{k_0, N\}$, by Lemma 4.12, that $\mathcal{P}_{n+1}(Z_{n+1}) = \mathcal{P}_n(Z_{n+1})$, hence by Lemma 4.13,

$$\omega^{n+1}(Z_{n+1}) \subseteq \mathcal{T} = \omega(\mathcal{T}') \implies \omega^n(Z_{n+1}) \subseteq \mathcal{T}'.$$

Therefore, \mathcal{T}' contains the patches $\omega^n(Z_{n+1}) \subseteq \mathcal{T}_1$ for all n sufficiently large, and these patches exhaust the entire tiling. Thus, $\mathcal{T}' = \mathcal{T}_1$, as desired. \square

5. INFINITE INVARIANT MEASURES.

5.1. Non-negative matrices revisited. We use the notation and results from Sections 2.5 and 2.6.

Definition 5.1. Let $M \in \mathcal{M}_{k \times k}(\mathbb{Z}_+)$. The infinite core of M is the set of all the vectors in $\text{core}_\infty(M) = \bigcap_{n \geq 1} M^n(\overline{\mathbb{R}}_+^k)$ where $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$.

We saw in Section 3 that $\text{core}(M)$ is isomorphic to the set of finite invariant measures. Here we will show that $\text{core}_\infty(M)$ is closely related to the set of “nice” invariant σ -finite measures, under some mild assumptions. The goal of this subsection is to describe the infinite core.

Lemma 5.2. Let M_1 and M_2 be two non-negative square matrices of dimensions n_1 and n_2 respectively, such that M_1 is primitive and M_2 has a positive eigenvalue $\rho_2 > 0$ associated to a positive eigenvector \mathbf{v}_2 . Let $C \neq 0$ be a non-negative $n_1 \times n_2$ -dimensional matrix and let ρ_1 be the Perron eigenvalue of M_1 . If there exists a vector $\mathbf{x} \in \overline{\mathbb{R}}_+^{n_1}$ such that

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{v}_2 \end{pmatrix} \text{ is in the infinite core of } M = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix},$$

then $\mathbf{x} = \infty$ whenever $\rho_1 \geq \rho_2$.

Proof. Let $n > 0$, C_n and $\mathbf{x}_n \in \overline{\mathbb{R}}_+^{n_1}$ be such that

$$\begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix}^n = \begin{pmatrix} M_1^n & C_n \\ 0 & M_2^n \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{v}_2 \end{pmatrix} = M^n \begin{pmatrix} \mathbf{x}_n \\ \frac{\mathbf{v}_2}{\rho_2^n} \end{pmatrix}.$$

Using the symbol “ \geq ” to denote the natural (component-wise) partial order on vector spaces, we have

$$(5.1) \quad \mathbf{x} = M_1^n \mathbf{x}_n + C_n \frac{\mathbf{v}_2}{\rho_2^n} \geq C_n \frac{\mathbf{v}_2}{\rho_2^n},$$

and

$$\begin{aligned} C_{n+1} \frac{\mathbf{v}_2}{\rho_2^{n+1}} &= \frac{1}{\rho_2^{n+1}} M_1^n C \mathbf{v}_2 + C_n \frac{\mathbf{v}_2}{\rho_2^n} \\ &= \frac{1}{\rho_2} \left(\frac{\rho_1}{\rho_2} \right)^n \left(\frac{M_1}{\rho_1} \right)^n C \mathbf{v}_2 + C_n \frac{\mathbf{v}_2}{\rho_2^n} \\ &= \frac{1}{\rho_2} \left[\sum_{k=0}^n \left(\frac{\rho_1}{\rho_2} \right)^k \left(\frac{M_1}{\rho_1} \right)^k \right] C \mathbf{v}_2. \end{aligned}$$

Thus if $\rho_1 \geq \rho_2$, we have

$$(5.2) \quad C_{n+1} \frac{\mathbf{v}_2}{\rho_2^{n+1}} \geq \frac{1}{\rho_2} \left[\sum_{k=0}^n \left(\frac{M_1}{\rho_1} \right)^k \right] C \mathbf{v}_2,$$

which tends to ∞ with n , because $C \mathbf{v}_2 \neq \mathbf{0}$ and $\lim_{n \rightarrow \infty} (M_1/\rho_1)^n = \mathbf{w} \mathbf{v} > 0$, where \mathbf{w} and \mathbf{v} are left and right Perron eigenvectors of M_1 respectively (see [14, Theorem 8.5.1]). Then from equations (5.1) and (5.2) we conclude that $\mathbf{x} = \infty$ when $\rho_1 \geq \rho_2$. \square

Suppose that $\alpha_1, \dots, \alpha_l$ are the equivalence classes associated to the matrix M . For every $1 \leq i \leq l$ we denote by M_i the restriction of M to the class α_i . We assume that M_i is primitive or equal to $[0]$. When M_i is primitive we denote by ρ_i the Perron eigenvalue of M_i ; if $M_i = [0]$ then $\rho_i = 0$.

Definition 5.3. Let M be a non-negative integer square matrix with irreducible components M_1, \dots, M_l . For every $1 \leq i \leq l$ such that M_i is primitive we consider

- (1) \mathcal{J}_i —the set of indices $j \in \{1, \dots, l\} \setminus \{i\}$ such that the class α_j has access to a class α_k , where α_k has access to α_i and $\rho_k \geq \rho_i$.
- (2) \mathcal{I}_i —the set containing i and all the indices $j \in \{1, \dots, l\} \setminus \mathcal{J}_i$ such that the class α_j has access to the class α_i (then necessarily $\rho_j < \rho_i$).

Remark 5.4. The classes α_j , with $j \in \mathcal{I}_i$ do not have access to the classes with indices in \mathcal{J}_i . The complement of $(\mathcal{I}_i \cup \mathcal{J}_i)$ is the set of j such that α_j does not have access to α_i .

Let $1 \leq i \leq l$ be such that M_i is primitive. The class α_i is distinguished with respect to $M|_{\mathcal{I}_i}$, the restriction of M to the set of indices in \mathcal{I}_i . Then Theorem 2.13 implies that there exists a unique $|\mathcal{I}_i|$ -dimensional normalized positive vector \mathbf{w}_i such that $M|_{\mathcal{I}_i} \mathbf{w}_i = \rho_i \mathbf{w}_i$. The restriction of \mathbf{w}_i to α_i is an eigenvector of M_i associated to ρ_i .

Definition 5.5. For every $1 \leq i \leq l$ such that M_i is primitive, we define \mathbf{y}_i in $\overline{\mathbb{R}}_+^k$ as follows:

- The restriction of \mathbf{y}_i to \mathcal{I}_i is equal to \mathbf{w}_i .
- The restriction of \mathbf{y}_i to \mathcal{J}_i is ∞ in every component.
- The restriction of \mathbf{y}_i to $(\mathcal{I}_i \cup \mathcal{J}_i)^c$ is zero.

For every $1 \leq i \leq l$, we define \mathbf{z}_i in $\overline{\mathbb{R}}_+^k$ as the vector whose restriction to $\mathcal{I}_i \cup \mathcal{J}_i$ is infinite, and \mathbf{z}_i restricted to $(\mathcal{I}_i \cup \mathcal{J}_i)^c$ is zero. If $M_i = [0]$ we define $\mathbf{y}_i = \mathbf{0}$.

Lemma 5.6. Let M be a non-negative integer $k \times k$ matrix with irreducible components M_1, \dots, M_l . For every $1 \leq i \leq l$ we have $\mathbf{y}_i \in \text{core}_\infty(M)$.

Proof. If $M_i = [0]$ then $\mathbf{y}_i = \mathbf{0}$ is in $\text{core}_\infty(M)$. Then we can assume that M_i is primitive.

For every $n \geq 1$, we define $\mathbf{y}_{n,i} \in \overline{\mathbb{R}}_+^k$, the vector such that

$$\mathbf{y}_{n,i}|_{\mathcal{I}_i} = \frac{\mathbf{w}_i}{\rho_i^n}, \quad \mathbf{y}_{n,i}|_{\mathcal{J}_i} = \infty, \quad \text{and} \quad \mathbf{y}_{n,i}|_{(\mathcal{I}_i \cup \mathcal{J}_i)^c} = \mathbf{0}.$$

For $1 \leq r \leq k$, the r -coordinate of $M^n \mathbf{y}_{n,i}$ is equal to

$$\sum_{s=1}^k M^n(r,s) \mathbf{y}_{n,i}(s) = \sum_{s \in \mathcal{I}_i} M^n(r,s) \mathbf{y}_{n,i}(s) + \sum_{s \in \mathcal{J}_i} M^n(r,s) \mathbf{y}_{n,i}(s).$$

Thus we have the following:

1. If $r \in (\mathcal{I}_i \cup \mathcal{J}_i)^c$, then $M^n(r,s) = 0$ for every $s \in \mathcal{I}_i \cup \mathcal{J}_i$. This implies the r -coordinate of $M^n \mathbf{y}_{n,i}$ is equal to $\mathbf{0} = \mathbf{y}_i(r)$.

2. If $r \in \mathcal{I}_i$, then $M^n(r,s) = 0$ for every $s \in \mathcal{J}_i$. This implies that the r -coordinate of $M^n \mathbf{y}_{n,i}$ is equal to

$$\sum_{s \in \mathcal{I}_i} M^n(r,s) \mathbf{y}_{n,i}(s) = \mathbf{w}_i(r) = \mathbf{y}_i(r).$$

3. If $r \in \mathcal{J}_i$, then for every $n \geq 1$ there exists $s_n \in \mathcal{J}_i$ such that $M^n(r,s_n) > 0$. This implies that r -coordinate of $M^n \mathbf{y}_{n,i}$ is equal to $M^n(r,s_n) \mathbf{y}_{n,i}(s_n) = \infty = \mathbf{y}_i(r)$.

The statements **1**, **2**, and **3** imply that $\mathbf{y}_i = M^n \mathbf{y}_{n,i}$, for every $n \geq 1$, which shows that \mathbf{y}_i is in the infinite core of M . \square

Lemma 5.7. *Let M be a non-negative integer $k \times k$ matrix with primitive irreducible components M_1, \dots, M_l . Then every vector $\mathbf{x} \in \text{core}_\infty(M)$ can be written as*

$$\mathbf{x} = \sum_{j=1}^l \lambda_j \mathbf{y}_j + \sum_{j=1}^l \delta_j \mathbf{z}_j,$$

where $\lambda_1, \dots, \lambda_l \geq 0$ and $\delta_1, \dots, \delta_l \in \{0, 1\}$. Conversely, every vector written in this way is in $\text{core}_\infty(M)$.

Proof. Let $\mathbf{x} \in \text{core}_\infty(M)$. Theorem 2.15 implies that when \mathbf{x} is finite, this vector is in the cone generated by the vectors \mathbf{y}_i , for every $1 \leq i \leq l$ such that α_i is distinguished for M . If \mathbf{x} has all its coordinates equal to ∞ or zero, it can be written as $\sum_{i=1}^l \delta_i \mathbf{z}_i$, for some $\delta_1, \dots, \delta_l \in \{0, 1\}$. Thus we can assume that \mathbf{x} has a finite positive coordinate and an infinite coordinate. For $1 \leq j \leq l$, let \mathbf{x}_j be the $|\alpha_j|$ -dimensional vector given by the restriction of \mathbf{x} to α_j . If some coordinate of \mathbf{x}_j is positive and finite, then all the coordinates of \mathbf{x}_j are positive and finite. Let $1 \leq i \leq l$ be such that

$$i = \max\{1 \leq k \leq l : 0 < \mathbf{x}_k < \infty\}.$$

If there exists $i < k \leq l$ such that $\mathbf{x}_k = \infty$ then for every $1 \leq j \leq l$ such that α_j has access to α_k we have $\mathbf{x}_j = \infty$. Then we can write

$$\mathbf{x} = \mathbf{u} + \sum_{j=1}^l \delta_j \mathbf{z}_j,$$

where $\mathbf{u}_j = \mathbf{x}_j$ for every $1 \leq j \leq i$, $\mathbf{u}_j = 0$ for every $i < j \leq l$, and $\delta_1, \dots, \delta_l \in \{0, 1\}$. After rearranging the coordinates of \mathbf{x} if it is necessary, we can suppose that there exists $1 \leq s \leq i$ such that $0 \leq \mathbf{x}_j < \infty$ for every $s \leq j \leq i$, and $\mathbf{x}_j = \infty$ for every $1 \leq l < s$. The vector $\mathbf{x}|_{\alpha_s, \dots, \alpha_i} = (\mathbf{x}_s, \dots, \mathbf{x}_i)$ is in the core of the restriction M' of M to the classes $\alpha_s, \dots, \alpha_i$. Then by Theorem 2.15, $\mathbf{x}|_{\alpha_s, \dots, \alpha_i}$ is in the cone generated by the distinguished eigenvectors of M' . Observe that \mathbf{v} is a distinguished eigenvector of M' if and only if \mathbf{v} is the restriction to $\alpha_s, \dots, \alpha_i$ of a scalar multiple of the vector \mathbf{y}_j , for some $s \leq j \leq i$ such that the class α_j is distinguished in M' . Thus, using Lemma 5.2, we can write

$$\mathbf{x} = \sum_{j=1}^l \lambda_j \mathbf{y}_j + \delta_j \mathbf{z}_j,$$

where $\lambda_1, \dots, \lambda_l \geq 0$ (with $\lambda_i > 0$), and $\delta_1, \dots, \delta_l \in \{0, 1\}$. The converse holds by Lemma 5.6. \square

5.2. Clopen nested partitions of the transversal. As in the previous sections, we consider a substitution ω defined on a set of prototiles $\mathcal{A} \subseteq \mathbb{R}^d$. We denote by $M \in \mathcal{M}_{\mathcal{A} \times \mathcal{A}}(\mathbb{Z}_+)$ the substitution matrix of ω , and $\mathcal{A}_1, \dots, \mathcal{A}_l$ the equivalence classes associated to M . We denote by M_i the restriction of M to the indices in \mathcal{A}_i , and we assume that M_i is primitive or equal to $[0]$. We suppose there are equivalence classes which are not associated to minimal components, namely, $\mathcal{A}_{m+1}, \dots, \mathcal{A}_l$, for some $1 \leq m < l$. We denote $\mathcal{A}' = \mathcal{A} \setminus (\mathcal{A}_1 \cup \dots \cup \mathcal{A}_m)$.

Recall that we denote by Γ the transversal of $X_{\mathcal{A}, \omega}$, and for every $A \in \mathcal{A}$, we set

$$C_A = \{\mathcal{T} \in \Gamma : A \in \mathcal{A}\}.$$

The collection $\mathcal{P}_0 = \{C_A : A \in \mathcal{A}\}$ is a clopen partition of Γ . For $n \geq 1$ and $A, B \in \mathcal{A}$, we define

$$\begin{aligned} D_{n,A} &= \text{supp}(\omega^n(A)), \\ J_{A,B}^{(n)} &= \{\mathbf{v} \in D_{n,A} : B + \mathbf{v} \subseteq \omega^n(A)\}, \\ J_A^{(n)} &= \bigcup_{B \in \mathcal{A}} J_{A,B}^{(n)}, \end{aligned}$$

and

$$\mathcal{P}_n = \{\omega^n(C_A) - \mathbf{v} : \mathbf{v} \in J_A^{(n)}, A \in \mathcal{A}\}.$$

(These \mathcal{P}_n have nothing to do with $\mathcal{P}_n(W)$ from Section 4.) **For the rest of this section we assume that the admissible substitution ω is partially recognizable, see Definition 4.3, and also**

(5.3) **For every prototile $A \in \mathcal{A}_{\text{nonp}}$, the patch $\omega(A)$ contains a tile of type $\mathcal{A}_{\text{nonp}}$.**

Note that the latter condition is satisfied if M has no components $[0]$, or if the non-periodic border condition holds.

Lemma 5.8. *For every $n \geq 0$, the collection \mathcal{P}_n is a covering of Γ . Furthermore,*

(i) For each $A \in \mathcal{A}$ and $n \geq 1$,

$$(5.4) \quad C_A = \bigcup_{B \in \mathcal{A}} \bigcup_{\mathbf{v} \in J_{B,A}^{(n)}} (\omega^n(C_B) - \mathbf{v}).$$

(ii) If $A, B \in \mathcal{A}$, $n \geq 1$, $\mathbf{v} \in J_A^{(n)}$ and $\mathbf{u} \in J_B^{(n)}$ are such that

$$(\omega^n(C_A) - \mathbf{v}) \cap (\omega^n(C_B) - \mathbf{u}) \neq \emptyset,$$

then $A = B$ and $\mathbf{v} = \mathbf{u}$, or $A, B \in \mathcal{A}_{\text{per}}$.

(iii) Let $\ell > n$, $A \in \mathcal{A}$, $B \in \mathcal{A}_{\text{nonp}}$, $\mathbf{v} \in J_A^{(\ell)}$ and $\mathbf{u} \in J_B^{(n)}$. We have

$$(\omega^\ell(C_A) - \mathbf{v}) \cap (\omega^n(C_B) - \mathbf{u}) \neq \emptyset,$$

if and only if $\omega^\ell(C_A) - \mathbf{v} \subseteq \omega^n(C_B) - \mathbf{u}$ and $B + \varphi^{-n}(\mathbf{v} - \mathbf{u}) \in \omega^{\ell-n}(A)$.

Proof. (i) Proposition 2.8 ensures that \mathcal{P}_n is a covering of Γ and implies (5.4).

(ii) By partial recognizability, if $A \neq B$ or $\mathbf{v} \neq \mathbf{u}$ then the tilings in $(\omega^n(C_A) - \mathbf{v}) \cap (\omega^n(C_B) - \mathbf{u})$ contain only tiles from \mathcal{A}_{per} . Hence the patches $\omega^n(A)$ and $\omega^n(B)$ only contain tiles in \mathcal{A}_{per} . Condition (5.3) implies that A and B are in \mathcal{A}_{per} .

(iii) Let $D_1, \dots, D_k \in \mathcal{A}$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$ be such that $\omega^{\ell-n}(A)$ is the disjoint union $\bigcup_{i=1}^k (D_i + \mathbf{x}_i)$. Then $\omega^\ell(A)$ is equal to the disjoint union $\bigcup_{i=1}^k (\omega^n(D_i) + \varphi^n(\mathbf{x}_i))$. This implies that there exists $1 \leq i \leq k$ such that $\mathbf{z} = \mathbf{v} - \varphi^n(\mathbf{x}_i) \in J_{D_i}^{(n)}$ and

$$\omega^\ell(C_A) - \mathbf{v} = \omega^\ell(C_A) - \varphi^n(\mathbf{x}_i) - \mathbf{z} \subseteq \omega^n(C_{D_i}) - \mathbf{z}.$$

Then by hypothesis we have $(\omega^n(C_B) - \mathbf{u}) \cap (\omega^n(C_{D_i}) - \mathbf{z}) \neq \emptyset$. Since $B \in \mathcal{A}_{\text{nonp}}$, from part (ii) it follows that $B = D_i$, $\mathbf{z} = \mathbf{u}$, and then $\omega^\ell(C_A) - \mathbf{v} \subseteq \omega^n(C_B) - \mathbf{u}$ and $B + \varphi^{-n}(\mathbf{v} - \mathbf{u}) \in \omega^{\ell-n}(A)$. The other direction of the equivalence is immediate. \square

Remark 5.9. 1. If the substitution is non-periodic, then $(\mathcal{P}_n)_{n \geq 0}$ is a nested sequence of clopen partitions of $X_{\mathcal{A},\omega}$. In the minimal non-periodic one-dimensional symbolic case, the sequences $(\mathcal{P}_n)_{n \geq 0}$ correspond to the sequence of Kakutani-Rohlin partitions for minimal substitution subshifts given in [9]. In general, it is only a covering, but we will see below that it becomes a partition if we intersect it with the set of non-periodic tilings.

2. It is useful to give an informal interpretation of the covering $(\mathcal{P}_n)_{n \geq 0}$. Given a tiling $\mathcal{T} \in \Gamma$, we have a sequence of tilings $(\mathcal{T}_n)_{n \geq 0}$, with $\mathcal{T}_0 = \mathcal{T}$, such that $\omega(\mathcal{T}_n) = \mathcal{T}_{n-1}$ for $n \geq 1$. This defines a sequence of “supertilings” obtained by composing the tiles of \mathcal{T} , with “supertiles” that are translates of $\omega^n(A)$ for $A \in \mathcal{A}$. This sequence is uniquely defined if \mathcal{T} contains a non-periodic tile. We consider the supertile of order n whose support contains the origin (it is uniquely defined since \mathcal{T} is in the transversal). This determines the element of \mathcal{P}_n to which \mathcal{T} belongs.

5.3. Necessary conditions for transverse measures.

Definition 5.10. Let μ^T be a transverse measure on $\mathcal{B}(\Gamma)$. For every $A \in \mathcal{A}$ and for every $n \geq 0$, we define

$$\mu_{n,A}^T = \mu^T(\omega^n(C_A) - \mathbf{v}),$$

where \mathbf{v} is a vector in $J_A^{(n)}$. The number $\mu_{n,A}^T$ does not depend on \mathbf{v} because μ is transverse. We denote by μ_n^T the vector $(\mu_{n,A}^T)_{A \in \mathcal{A}}$ and by $\tilde{\mu}_n^T$ the vector $(\mu_{n,A}^T)_{A \in \mathcal{A}'}$.

Lemma 5.11. *Let μ be a transverse measure on $\mathcal{B}(\Gamma)$. Then for every $A \in \mathcal{A}'$ and $\ell > n \geq 0$,*

$$\mu_{n,A}^T = \sum_{B \in \mathcal{A}'} M^{\ell-n}(A, B) \mu_{\ell,B}^T.$$

Thus,

$$\tilde{\mu}_n^T = (M')^{\ell-n} \tilde{\mu}_{\ell}^T,$$

where M' is the restriction of M to the set of indices in \mathcal{A}' .

Proof. Let $A \in \mathcal{A}'$ and $\mathbf{u} \in J_A^{(n)}$. From (iii) of Lemma 5.8 we have

$$\omega^n(C_A) - \mathbf{u} = \bigcup_{B \in \mathcal{A}} \bigcup_{\mathbf{v} \in I_B^{(n)}} (\omega^\ell(C_B) - \mathbf{v}),$$

where $I_B^{(n)}$ is the set of \mathbf{v} in $J_B^{(n)}$ such that $A + \varphi^{-n}(\mathbf{v} - \mathbf{u}) \in \omega^{\ell-n}(B)$. Since the minimal components are ω -invariant we can restrict the outer union to \mathcal{A}' :

$$\omega^n(C_A) - \mathbf{u} = \bigcup_{B \in \mathcal{A}'} \bigcup_{\mathbf{v} \in I_B^{(n)}} (\omega^\ell(C_B) - \mathbf{v}).$$

Observe that $|I_B^{(n)}| = M^{\ell-n}(A, B)$. Thus from (ii) of Lemma 5.8 we obtain the desired equality. \square

Remark 5.12. If $\mathcal{A}_{per} = \emptyset$ then the same proof shows that

$$\mu_{n,A}^T = \sum_{B \in \mathcal{A}} M^{\ell-n}(A, B) \mu_{\ell,B}^T,$$

for every $A \in \mathcal{A}$ and $\ell > n \geq 0$. It follows that $\mu_0^T = M^n \mu_n^T$ for every $n > 0$, hence this vector belongs to $core_\infty(M)$. Thus Lemmas 5.2, 5.7 and 5.11 imply that if $\mu_{0,A}^T > 0$ for some $A \in \mathcal{A}'$, then the restriction of μ_0^T to every minimal component which has access to A is infinity (because the component of A is not distinguished). In the next lemma we show the same for the general case.

Lemma 5.13. *Let μ^T be a transverse measure on $\mathcal{B}(\Gamma)$ and let $1 \leq p \leq m$. If there is $A \in \mathcal{A}_p$ for which there exist $D \in \mathcal{A}'$ and $n > 0$ verifying $\mu^T(C_D) > 0$ and $M^n(A, D) > 0$, then $\mu^T(C_E) = \infty$, for every $E \in \mathcal{A}_p$.*

Proof. Let

$$X_{\text{nonp}} := \{\mathcal{T} \in X_{\mathcal{A}, \omega} : \mathcal{T} \text{ contains a tile of type } \mathcal{A}_{\text{nonp}}\}.$$

This set is open and invariant under the \mathbb{R}^d translation action. Using partial recognizability, we obtain, exactly as in the proof of Lemma 5.8, the following statements:

(ii') If $A, B \in \mathcal{A}$, $n \geq 1$, $\mathbf{v} \in J_A^{(n)}$ and $\mathbf{u} \in J_B^{(n)}$ are such that

$$(\omega^n(C_A) - \mathbf{v}) \cap (\omega^n(C_B) - \mathbf{u}) \cap X_{\text{nonp}} \neq \emptyset,$$

then $A = B$ and $\mathbf{v} = \mathbf{u}$.

(iii') Let $\ell > n$, $A, B \in \mathcal{A}$, $\mathbf{v} \in J_A^{(\ell)}$ and $\mathbf{u} \in J_B^{(n)}$. We have

$$(\omega^\ell(C_A) - \mathbf{v}) \cap (\omega^n(C_B) - \mathbf{u}) \cap X_{\text{nonp}} \neq \emptyset,$$

if and only if $\omega^\ell(C_A) - \mathbf{v} \subseteq \omega^n(C_B) - \mathbf{u}$ and $B + \varphi^{-n}(\mathbf{v} - \mathbf{u}) \in \omega^{\ell-n}(A)$.

In other words, by intersecting \mathcal{P}_n with X_{nonp} we recover the nested partition properties even in the presence of periodic minimal components.

Next we can argue as in Lemma 5.11 to deduce that the vector

$$(\mu^T(\omega^n(C_A) \cap X_{\text{nonp}}))_{A \in \mathcal{A}}$$

belongs to the infinite core of M for every $n \geq 0$. Let $D \in \mathcal{A}_i \subseteq \mathcal{A}'$ be such that $\mu(C_D) > 0$.

Then $D \in \mathcal{A}_{\text{nonp}}$, so

$$\mu^T(C_D \cap X_{\text{nonp}}) = \mu^T(C_D) > 0.$$

The assumption of the lemma implies that the class \mathcal{A}_p has access to \mathcal{A}_i , and since \mathcal{A}_p corresponds to a minimal component, we have $\rho_p > \rho_i$. Now it follows by Lemma 5.7 that

$$\mu^T(C_E) \geq \mu^T(C_E \cap X_{\text{nonp}}) = \infty \quad \text{for all } E \in \mathcal{A}_p.$$

□

5.4. Constructing infinite transverse measures. In Section 3 we proved that finite invariant measures of the substitution tiling system $(X_{\mathcal{A}, \omega}, \mathbb{R}^d) = (X, \mathbb{R}^d)$ are supported on its minimal components. Therefore, if μ is a finite invariant measure and μ^T is the associated transverse measure, then $\mu^T(C_D) = 0$ for every prototile $D \in \mathcal{A}'$. In this section we characterize the infinite σ -finite invariant measures μ for which there exists $D \in \mathcal{A}'$ such that $0 < \mu^T(C_D) < \infty$. It follows from Lemmas 5.11 and 5.13 that the values of μ^T on the elements of \mathcal{P}_n belong to the infinite core of M (at least, if we exclude periodic components). Lemma 5.7 suggests that those which correspond to ergodic measures should come from the vectors \mathbf{y}_i . We will show that this is indeed the case, under some mild assumptions.

Recall that $\mathcal{A}_1, \dots, \mathcal{A}_m$ are the equivalence classes of M associated to the minimal components, and $\mathcal{A}' = \mathcal{A} \setminus \cup_{i=1}^m \mathcal{A}_i = \mathcal{A}_{m+1} \cup \dots \cup \mathcal{A}_l$. Let $m+1 \leq i \leq l$ be such that M_i is primitive. Let $\mathcal{A}_{\mathcal{J}_i^c}$ be the set of prototiles $A \in \mathcal{A}$ such that if $A \in \mathcal{A}_j$ then $j \in \mathcal{J}_i^c$ (see Definition 5.3 for \mathcal{J}_i). Equivalently, $\mathcal{A}_{\mathcal{J}_i^c}$ is the set of prototiles with indices in \mathcal{I}_i and those which have no access to the class \mathcal{A}_i . By definition, \mathcal{A}_i is a distinguished class for the restriction of M to the indices in $\mathcal{A}_{\mathcal{J}_i^c}$. A class \mathcal{A}_j , with $j \leq m$, corresponding to a minimal component, is in $\mathcal{A}_{\mathcal{J}_i^c}$ if and only if it has no access to \mathcal{A}_i . We define

$$\Gamma_i = \bigcup_{n \geq 0} \bigcup_{A \in \mathcal{A}_{\mathcal{J}_i^c}} \bigcup_{\mathbf{v} \in J_A^{(n)}} (\omega^n(C_A) - \mathbf{v}),$$

and \mathcal{F}_i , the collection of subsets of Γ_i given by

$$\mathcal{F}_i = \{\omega^n(C_A) - \mathbf{v} : A \in \mathcal{A}_{\mathcal{J}_i^c}, \mathbf{v} \in J_A^{(n)}, n \geq 0\}.$$

Recalling the informal description of partition elements from Remark 5.9, we note that the tilings in Γ_i are those for which the “supertiles” of order n containing the origin are of “type”

$A_{\mathcal{J}_i^c}$ for n sufficiently large. Observe that if this is the case for some n_0 , then this is also true for $n > n_0$, since if a tile of type \mathcal{A}_j occurs in $\omega^n(B)$ for $B \in \mathcal{A}_k$, then j has access to k , and so either $j = k$, or k does not have access to j . Let $\mathbf{y}_i \in \overline{\mathbb{R}}_+^\mathcal{A}$ be the vector given in Definition 5.5 for the class \mathcal{A}_i . For $n \geq 0$, let $\mathbf{y}_{n,i} \in \overline{\mathbb{R}}_+^\mathcal{A}$ be such that $\mathbf{y}_i = M^n \mathbf{y}_{n,i}$ (that is, $\mathbf{y}_{n,i} = \mathbf{y}_i / \rho_i^n$, where ρ_i is the Perron eigenvalue of M_i). We define the function $\phi_i : \mathcal{F}_i \cup \{\emptyset\} \rightarrow \mathbb{R}_+$ by $\phi_i(\emptyset) = 0$ and

$$(5.5) \quad \phi_i(\omega^n(C_A) - \mathbf{v}) = \mathbf{y}_{n,i}(A), \text{ for every } A \in A_{\mathcal{J}_i^c} \text{ and } n \geq 0.$$

Since $\mathbf{y}_i|_{\mathcal{J}_i^c} < \infty$, this function is well-defined. A standard argument shows that the function $\phi_i^* : 2^{\Gamma_i} \rightarrow \overline{\mathbb{R}}_+$ given by

$$\phi_i^*(U) = \inf \left\{ \sum_{n \in \mathbb{N}} \phi_i(C_n) : (C_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_i \cup \{\emptyset\}, U \subseteq \bigcup_{n \in \mathbb{N}} C_n \right\}$$

is an outer measure. Observe that ϕ_i^* is well-defined because \mathcal{F}_i is countable and the union of all the sets in \mathcal{F}_i is equal to Γ_i . The collection

$$\eta_i^* = \{U \subseteq \Gamma_i : \forall E \subseteq \Gamma_i, \phi_i^*(E) \geq \phi_i^*(E \cap U) + \phi_i^*(E \setminus U)\}$$

is a σ -algebra and the restriction of ϕ_i^* to η_i^* is a complete measure (every negligible set with respect to ϕ_i^* is in η_i^*).

Lemma 5.14. $\mathcal{F}_i \subseteq \eta_i^*$.

Proof. Let $U = \omega^n(C_A) - \mathbf{v} \in \mathcal{F}_i$, with $A \in A_{\mathcal{J}_i^c}$ and $\mathbf{v} \in J_A^{(n)}$. We need to show

$$(5.6) \quad \phi_i^*(E) \geq \phi_i^*(E \cap U) + \phi_i^*(E \setminus U)$$

for $E \subseteq \Gamma_i$.

Step 1. Suppose first that $E = \omega^m(C_B) - \mathbf{w}$ is also in \mathcal{F}_i . If A (resp. B) is in a minimal component, then $\phi_i(U) = 0$ (resp. $\phi_i(E) = 0$). This immediately implies (5.6). Thus we can assume that A and B are in $\mathcal{A}_{\text{nonp}}$. The inequality (5.6) is also clear if $U \cap E = \emptyset$ or $E \setminus U = \emptyset$.

If $m \geq n$, then $U \cap E \neq \emptyset$ implies that $E \subseteq U$ by Lemma 5.8(iii); then $E \setminus U = \emptyset$ and we are done.

If $m < n$ and $U \cap E \neq \emptyset$, then $U \cap E = U$. In this case we have

$$E \setminus U = \bigcup_{D \in A_{\mathcal{J}_i^c}} \bigcup_{\substack{\mathbf{u} \in J_D^{(n)} \\ \omega^n(C_D) - \mathbf{u} \subseteq E \\ \mathbf{u} \neq \mathbf{v} \text{ if } D = A}} (\omega^n(C_D) - \mathbf{u}),$$

which implies that

$$\begin{aligned} \phi_i^*(E \setminus U) &\leq \sum_{D \in A_{\mathcal{J}_i^c}} \sum_{\substack{\mathbf{u} \in J_D^{(n)} \\ \omega^n(C_D) - \mathbf{u} \subseteq E \\ \mathbf{u} \neq \mathbf{v} \text{ if } D = A}} \phi_i^*(\omega^n(C_D) - \mathbf{u}) \\ &= \sum_{D \in A_{\mathcal{J}_i^c}} M^{n-m}(B, D) \mathbf{y}_{n,i}(D) - \phi_i^*(U). \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_i^*(E) = \mathbf{y}_{m,i}(B) &= \sum_{D \in A_{\mathcal{J}_i^c}} M^{n-m}(B, D) \mathbf{y}_{n,i}(D) \\ &\geq \phi_i^*(E \setminus U) + \phi_i^*(U) = \phi_i^*(E \setminus U) + \phi_i^*(U \cap E), \end{aligned}$$

as desired.

Step 2. If E is any subset of Γ_i , then we have two cases:

- (a) if $\phi_i^*(E) = \infty$, then (5.6) is clear.
- (b) If $\phi_i^*(E) < \infty$, then given $\varepsilon > 0$, there exists $(U_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_i$ such that

$$\sum_{n \in \mathbb{N}} \phi_i^*(U_n) \leq \phi_i^*(E) + \varepsilon.$$

Using Step 1 and the fact that ϕ_i^* is an outer measure, we get

$$\begin{aligned} \phi_i^*(E \cap U) + \phi_i^*(E \setminus U) &\leq \phi_i^*\left(\bigcup_{n \in \mathbb{N}} U_n \cap U\right) + \phi_i^*\left(\bigcup_{n \in \mathbb{N}} U_n \setminus U\right) \\ &\leq \sum_{n \in \mathbb{N}} (\phi_i^*(U_n \cap U) + \phi_i^*(U_n \setminus U)) \\ &\leq \sum_{n \in \mathbb{N}} \phi_i^*(U_n) \\ &\leq \phi_i^*(E) + \varepsilon, \end{aligned}$$

concluding the proof. \square

In the sequel we will prove that under some conditions, the Borel sets of Γ_i (with respect to the induced topology) are contained in η^* . We define

$$Y_i = \{\mathcal{T} \in X_{\mathcal{A}, \omega} : \mathcal{T} \text{ has a tile equivalent to some } A \in \mathcal{A}_i\}$$

and

$$\tilde{\Gamma}_i = \bigcup_{m \geq 0} \bigcap_{n \geq m} \bigcup_{A \in \mathcal{A}_i} \bigcup_{\mathbf{v} \in J_A^{(n)}} (\omega^n(C_A) - \mathbf{v}).$$

Recall (Section 2.4) that in the special case when \mathcal{A}_i is a maximal irreducible component of the graph $G(M^T)$, the set Y_i is a maximal component of the tiling space; here we consider the same kind of set for an arbitrary non-minimal component.

Lemma 5.15. *We have*

- (i) $\phi_i^*(\Gamma_i \setminus \tilde{\Gamma}_i) = 0$;
- (ii) $\phi_i^*(\Gamma_i \setminus Y_i) = 0$.

Proof. Note that (ii) follows from (i). Indeed, $\mathcal{T} \in \tilde{\Gamma}_i$ if and only if eventually all “supertiles” containing the origin (see Remark 5.9) have a type from \mathcal{A}_i . Since we assumed that $M_i \neq [0]$, any substitution of a tile in \mathcal{A}_i must contain a tile in \mathcal{A}_i , so $\tilde{\Gamma}_i \subseteq Y_i$.

It remains to verify (i). For every $j \in \mathcal{J}_i^c$ and $m \geq 0$, we define

$$\Gamma_{i,j,m} = \bigcap_{n \geq m} \bigcup_{A \in \mathcal{A}_j} \bigcup_{\mathbf{v} \in J_A^{(n)}} (\omega^n(C_A) - \mathbf{v}) \quad \text{and} \quad \Gamma_{i,j} = \bigcup_{m \geq 0} \Gamma_{i,j,m}.$$

We have

$$(5.7) \quad \Gamma_i \setminus \tilde{\Gamma}_i = \bigcup_{j \in \mathcal{J}_i^c \setminus \{i\}} \Gamma_{i,j}.$$

Indeed, $\Gamma_{i,j}$ is the set of tilings for which the supertiles containing the origin are eventually of type from \mathcal{A}_j . Since we assumed that all non-zero components of M are primitive, the types of the supertiles containing the origin must stabilize into types from one of the components, hence the claim (5.7). Let $j \in \mathcal{J}_i^c$, $j \neq i$. If j has no access to i , then the definition of ϕ_i^* implies that $\phi_i^*(\Gamma_{i,j}) = 0$. If $M_j = [0]$ then $\Gamma_{i,j} = \emptyset$. Thus we can assume that $j \in \mathcal{I}_i$ and M_j is primitive. Let $A \in \mathcal{A}_j$, $m \geq 0$ and $\mathbf{v} \in J_A^{(m)}$. For every $n \geq m$ we have

$$\Gamma_{i,j,m} \cap (\omega^m(C_A) - \mathbf{v}) \subseteq \bigcup_{B \in \mathcal{A}_j} \bigcup_{\substack{\mathbf{w} \in J_B^{(n)} \\ \omega^n(C_B) - \mathbf{w} \subseteq \omega^m(C_A) - \mathbf{v}}} (\omega^n(C_B) - \mathbf{w}),$$

which implies that

$$\begin{aligned} \phi_i^*(\Gamma_{i,j,m} \cap (\omega^m(C_A) - \mathbf{v})) &\leq \sum_{B \in \mathcal{A}_j} M^{n-m}(A, B) \frac{\mathbf{y}_i(B)}{\rho_i^n} \\ &= \rho_j^{-m} \sum_{B \in \mathcal{A}_j} \frac{M_j^{n-m}(A, B)}{\rho_j^{n-m}} \left(\frac{\rho_j}{\rho_i}\right)^n \mathbf{y}_i(B) \end{aligned}$$

Since $\lim_{n \rightarrow \infty} (M_j^{n-m}(A, B) / \rho_j^{n-m})$ exists and is finite, and since $\rho_j < \rho_i$, we get

$$\phi_i^*(\Gamma_{i,j,m} \cap (\omega^m(C_A) - \mathbf{v})) = 0.$$

This implies that $\phi_i^*(\Gamma_{i,j,m}) = 0$, and then $\phi_i^*(\Gamma_i \setminus \tilde{\Gamma}_i) = 0$. \square

For $n \geq 0$ and $A \in \mathcal{A}$ we define

$$I_{n,r,A} = \bigcup_{B \in \mathcal{A}} \{\mathbf{v} \in J_{A,B}^{(n)} : (\text{supp}(B) + \mathbf{v}) \cap (\partial D_{n,A})^{+r} \neq \emptyset\}.$$

In other words, $I_{n,r,A}$ is the set of \mathbf{v} such that $B + \mathbf{v}$ occurs in $\omega^n(A)$ for some B , within distance r from the boundary. It is straightforward to show that the tilings $\mathcal{T} \in \Gamma_i$ for which

there exists $\mathbf{v} \in \mathbb{R}^d$ such that $\mathcal{T} - \mathbf{v} \in \Gamma \setminus \Gamma_i$ are in

$$(5.8) \quad C = \bigcup_{r \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} \bigcup_{A \in A_{\mathcal{J}_i^c}} \bigcup_{\mathbf{v} \in I_{n,r,A}} (\omega^n(C_A) - \mathbf{v}).$$

The set C is a “bad set” for us: it is the set of tilings in Γ_i which “belong to the border” in the following sense: the union of supertiles containing the origin, discussed in Remark 5.9, is not the entire space \mathbb{R}^d . Observe that

$$(5.9) \quad \mathcal{T} \in \Gamma_i \setminus C \implies \bigcap_{n \in \mathbb{N}} (\omega^n(C_{A_n}) - \mathbf{v}_n) = \{\mathcal{T}\},$$

where $\omega^n(C_{A_n}) - \mathbf{v}_n \in \mathcal{P}_n$ is the set containing \mathcal{T} , for every $n \in \mathbb{N}$. Note also that C is translation-invariant. The next lemma gives a sufficient condition for C to be negligible with respect to ϕ_i^* .

Lemma 5.16. *If there exist $D \in \mathcal{A}_i$ and $n > 0$ such that a translate of D appears in the interior of $\omega^n(D)$, then C is negligible with respect to ϕ_i^* .*

Proof. In view of Lemma 5.15, it is enough to show that for every $m \geq 0$,

$$C_{i,m} := C \cap \bigcap_{n \geq m} \bigcup_{A \in \mathcal{A}_i} \bigcup_{\mathbf{v} \in J_A^{(n)}} (\omega^n(C_A) - \mathbf{v})$$

is negligible with respect to ϕ_i^* . It is clear that

$$C_{i,m} = \bigcup_{r \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{A \in \mathcal{A}_i} \bigcup_{\mathbf{v} \in I_{n,r,A}} (\omega^n(C_A) - \mathbf{v}).$$

We have $C_{i,m} = \bigcup_{r \in \mathbb{N}} \bigcap_{n \geq m} C_{i,n,r}$ where

$$C_{i,n,r} = \bigcup_{A \in \mathcal{A}_i} \bigcup_{\mathbf{v} \in I_{n,r,A}} (\omega^n(C_A) - \mathbf{v}).$$

Fix $r \in \mathbb{N}$. It is enough to show that $\phi_i^*(\bigcap_{n \geq m} C_{i,n,r}) = 0$ for m sufficiently large, and we will do this by estimating $\phi_i^*(C_{i,n,r} \cap C_{i,m,r})$ for $n > m$. For $A \in \mathcal{A}_i$ consider the decomposition of $\omega^n(A)$ into supertiles of order m , which is the inflated decomposition of $\omega^{n-m}(A)$ into tiles. By the “border” of $\omega^{n-m}(A)$ we mean the patch of tiles whose supports intersect the boundary of $\text{supp}(\omega^{n-m}(A))$. Applying ω^m to the border increases its width, hence we can choose m sufficiently large, so that any tile in $\omega^n(A)$ within distance r from $\partial D_{n,A} = \partial(\text{supp}(\omega^n(A)))$ belongs to a supertile $\omega^m(B)$ in the inflated border. Then we have

$$\phi_i^*(C_{i,n,r} \cap C_{i,m,r}) \leq \sum_{A \in \mathcal{A}_i} \sum_{B \in \mathcal{A}_i} b_{n-m,B,A} |I_{m,r,B}| \frac{\mathbf{y}_i(A)}{\rho_i^n},$$

where $b_{k,B,A}$ is the number of different translates of B which appear in the border of $\omega^k(A)$.

On the other hand, the hypothesis implies that there exists $n_0 > 0$ such that for every $A, B \in \mathcal{A}_i$ there exists a translate of B in the interior of $\omega^{n_0}(A)$. Thus, there exists $0 \leq \delta < 1$

such that $b_{n_0, B, A} \leq \delta M^{n_0}(B, A)$ for every $A, B \in \mathcal{A}_i$. Inductively, we deduce $b_{kn_0, B, A} \leq \delta^k M^{kn_0}(B, A)$ for every $k > 0$ and for every $A, B \in \mathcal{A}_i$. Hence we get

$$\begin{aligned}\phi_i^*(C_{i,m+nn_0,r} \cap C_{i,m,r}) &\leq \delta^n \sum_{A \in \mathcal{A}_i} \sum_{B \in \mathcal{A}_i} M^{nn_0}(B, A) |I_{m,r,B}| \frac{\mathbf{y}_i(A)}{\rho_i^{m+nn_0}} \\ &= \delta^n \sum_{B \in \mathcal{A}_i} |I_{m,r,B}| \frac{\mathbf{y}_i(B)}{\rho_i^m},\end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \phi_i^*(C_{i,m+nn_0,r} \cap C_{i,m,r}) = 0$ and then $\phi_i^*(C) = 0$. \square

In the sequel we will suppose that the hypothesis of Lemma 5.16 holds. That is, we will assume that

$$(5.10) \quad \exists A \in \mathcal{A}_i, \exists n > 0 \text{ such that a translate of } A \text{ appears in the interior of } \omega^n(A).$$

Remark 5.17. If \mathcal{A}_i is one of the maximal components, that is, $A \in \mathcal{A}_i$ does not appear in the substitution of any prototile from another component, then (5.10) holds automatically, because the admissibility assumption implies that A must appear in the interior of $\omega^n(E)$ for some tile E , which can only be from \mathcal{A}_i . Lemma 5.16 implies that in this case, the set C is always negligible with respect to ϕ_i^* .

Lemma 5.18. Suppose that ω verifies (5.10). Then for every non-empty open set $U \subseteq \Gamma_i$ there exists a countable collection $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_i$ of disjoint sets such that $\bigcup_{n \in \mathbb{N}} C_n \subseteq U$ and $\phi_i^*(U \setminus \bigcup_{n \in \mathbb{N}} C_n) = 0$. Thus $\mathcal{B}(\Gamma_i) \subseteq \eta_i^*$.

Proof. Let $U_1 \subseteq U$ the set of all $\mathcal{T} \in U$ for which there exist $n_{\mathcal{T}} \in \mathbb{N}$, $A_{\mathcal{T}} \in \mathcal{A}_{\mathcal{J}_i^c}$ and $\mathbf{v}_{\mathcal{T}} \in J_{A_{\mathcal{T}}}^{(n_{\mathcal{T}})}$ such that

$$\mathcal{T} \in \omega^{n_{\mathcal{T}}}(C_{A_{\mathcal{T}}}) - \mathbf{v}_{\mathcal{T}} \subseteq U.$$

Note that $U \setminus U_1 \subseteq C$ by (5.9). We have

$$U_1 \subseteq \bigcup_{\mathcal{T} \in U_1} \omega^{n_{\mathcal{T}}}(C_{A_{\mathcal{T}}}) - \mathbf{v}_{\mathcal{T}} \subseteq U,$$

and since the collection \mathcal{F}_i is countable, there exists a sequence $(\mathcal{T}_n)_{n \in \mathbb{N}} \subseteq U_1$ such that

$$U_1 \subseteq \bigcup_{\mathcal{T} \in U_1} (\omega^{n_{\mathcal{T}}}(C_{A_{\mathcal{T}}}) - \mathbf{v}_{\mathcal{T}}) = \bigcup_{k \in \mathbb{N}} \left(\omega^{n_{\mathcal{T}_k}}(C_{A_{\mathcal{T}_k}}) - \mathbf{v}_{\mathcal{T}_k} \right) \subseteq U.$$

Moreover, thanks to Lemma 5.8, the sets $(\omega^{n_{\mathcal{T}_k}}(C_{A_{\mathcal{T}_k}}) - \mathbf{v}_{\mathcal{T}_k})_{k \in \mathbb{N}}$ can be chosen disjoint. Since

$$U \setminus \bigcup_{k \in \mathbb{N}} \left(\omega^{n_{\mathcal{T}_k}}(C_{A_{\mathcal{T}_k}}) - \mathbf{v}_{\mathcal{T}_k} \right) \subseteq U \setminus U_1 \subseteq C,$$

Lemma 5.16 implies that U is in η_i^* , because it is a countable union of sets in \mathcal{F}_i up to a negligible set with respect to ϕ_i^* . \square

Now define $\mu_i^T : \mathcal{B}(\Gamma) \rightarrow \overline{\mathbb{R}}_+$ by

$$\mu_i^T(U) = \phi_i^*(U \cap \Gamma_i) \text{ for every } U \in \mathcal{B}(\Gamma).$$

Lemma 5.19. *Suppose that ω verifies (5.10). Then μ_i^T is a σ -finite transverse measure on $\mathcal{B}(\Gamma)$ supported on $Y_i \cap \Gamma$. Furthermore,*

$$\mu_i^T(C_A) = \mathbf{y}_i(A) \text{ for every } A \in \mathcal{A}.$$

Proof. Lemma 5.18 ensures that the restriction of ϕ_i^* to $\mathcal{B}(\Gamma_i)$ is a measure. Then μ_i^T is a measure on $\mathcal{B}(\Gamma)$. It is σ -finite because $\mu_i^T(\Gamma \setminus \Gamma_i) = 0$ and Γ_i is a countable union of sets of finite measure. Lemma 5.15 implies that μ_i^T is supported on $Y_i \cap \Gamma$.

Next we prove that μ_i^T is transverse. Let $U \in \mathcal{B}(\Gamma)$ and $\mathbf{v} \in \mathbb{R}^d$ be such that $U - \mathbf{v} \subseteq \Gamma$.

Step 1. First we suppose that $U = \omega^n(C_A) - \mathbf{u}$, for some $A \in A_{\mathcal{J}_i^c} \cap \mathcal{A}'$, $\mathbf{u} \in J_A^{(n)}$ and $n \geq 0$. If $\mathbf{u} + \mathbf{v} \in J_A^{(n)}$ then by definition of μ_i^T we have $\mu_i^T(U) = \mu_i^T(U - \mathbf{v})$. If not, for $m > n$ consider the sets $\omega^m(C_{A_{m,1}}) - \mathbf{u}_{m,1}, \dots, \omega^m(C_{A_{m,k_m}}) - \mathbf{u}_{m,k_m}$ in \mathcal{F}_i whose union is equal to U (this corresponds to looking at m -level supertiles and finding translates of $\omega^n(A)$ in them). This union is disjoint since $A \in \mathcal{A}'$. Let $J_m = \{1 \leq i \leq k_m : u_{m,i} + \mathbf{v} \in J_{A_{m,i}}^{(m)}\}$ and $U_m = \bigcup_{i \notin J_m} (\omega^m(C_{A_{m,i}}) - \mathbf{u}_{m,i})$. We have

$$\begin{aligned} \mu_i^T(U) &= \sum_{i \in J_m} \mu_i^T(\omega^m(C_{A_{m,i}}) - \mathbf{u}_{m,i}) + \mu_i^T(U_m), \\ \mu_i^T(U - \mathbf{v}) &= \sum_{i \in J_m} \mu_i^T(\omega^m(C_{A_{m,i}}) - \mathbf{u}_{m,i}) + \mu_i^T(U_m - \mathbf{v}), \end{aligned}$$

hence

$$\mu_i^T(U) - \mu_i^T(U - \mathbf{v}) = \mu_i^T(U_m) - \mu_i^T(U_m - \mathbf{v}).$$

Note that $U_{m+1} \subseteq U_m$ and $\bigcap_m U_m \subseteq C$, $\bigcap_m (U_m - \mathbf{v}) \subseteq C$, so Lemma 5.16 implies $\mu_i^T(U - \mathbf{v}) = \mu_i^T(U)$. If $A \in A_{\mathcal{J}_i^c} \setminus \mathcal{A}'$, then A is in a minimal component which has no access to \mathcal{A}_i and we have $\mu_i^T(U) = 0$. Then a similar argument yields $\mu_i^T(U - \mathbf{v}) \leq \mu_i^T(U_m - \mathbf{v})$, whence $\mu_i^T(U - \mathbf{v}) = 0$.

Step 2. Now we suppose that $U \subseteq \Gamma_i$ is an open set. Let $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_i$ be a disjoint collection of sets such that $\bigcup_{n \in \mathbb{N}} C_n \subseteq U$ and $\mu_i^T(U) = \sum_{n \in \mathbb{N}} \mu_i^T(C_n)$. This collection exists due to Lemma 5.18. On the one hand, Step 1 implies that $\mu_i^T(U) = \sum_{n \in \mathbb{N}} \mu_i^T(C_n - \mathbf{v}) = \mu_i^T(\bigcup_{n \in \mathbb{N}} (C_n - \mathbf{v}))$. On the other hand, $(U - \mathbf{v}) \setminus \bigcup_{n \in \mathbb{N}} (C_n - \mathbf{v}) \subseteq C$ which implies that $\mu_i^T(U - \mathbf{v}) = \mu_i^T(U)$.

Step 3. Now let U be any set in $\mathcal{B}(\Gamma)$. Since the elements in \mathcal{F}_i are clopen sets in Γ_i , we have

$$\begin{aligned} \phi_i^*(U \cap \Gamma_i) &= \inf \left\{ \sum_{n \in \mathbb{N}} \mu_i^T(C_n) : (C_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_i \cup \emptyset, U \cap \Gamma_i \subseteq \bigcup_{n \in \mathbb{N}} C_n \right\} \\ (5.11) \quad &\geq \inf \{ \mu_i^T(V) : U \cap \Gamma_i \subseteq V, \text{ and } V \text{ is open in } \Gamma_i \}. \end{aligned}$$

Let V be an open set in Γ_i which contains $U \cap \Gamma_i$. We have

$$\mu_i^T((U \cap \Gamma_i) - \mathbf{v}) \leq \mu_i^T(V - \mathbf{v}) = \mu_i^T(V),$$

which implies, by (5.11),

$$\mu_i^T((U \cap \Gamma_i) - \mathbf{v}) \leq \phi_i^*(U \cap \Gamma_i) = \mu_i^T(U).$$

We claim that $\mu_i^T((U \cap \Gamma_i) - \mathbf{v}) = \mu_i^T(U - \mathbf{v})$. Indeed,

$$\mu_i^T(U - \mathbf{v}) = \mu_i^T((U \cap \Gamma_i) - \mathbf{v}) + \mu_i^T((U \cap \Gamma_i^c) - \mathbf{v}),$$

but

$$\mu_i^T((U \cap \Gamma_i^c) - \mathbf{v}) = \mu_i^T(((U \cap \Gamma_i^c) - \mathbf{v}) \cap \Gamma_i) = 0,$$

because $((U \cap \Gamma_i^c) - \mathbf{v}) \cap \Gamma_i \subseteq C$. The claim is proved, and we obtain $\mu_i^T(U - \mathbf{v}) \leq \mu_i^T(U)$.

Finally, replacing U by $U - \mathbf{v}$ and \mathbf{v} by $-\mathbf{v}$ we get $\mu_i^T(U) = \mu_i^T(U - \mathbf{v})$. This concludes the proof that μ_i^T is a transverse measure.

It remains to verify the formula. By definition, $\mu_i^T(C_A) = \mathbf{y}_i(A)$ for every $A \in A_{\mathcal{J}_i^c}$. If $A \in \mathcal{J}_i$ then Lemma 5.11, Lemma 5.2 and Lemma 5.13 imply that $\mu_i^T(C_A) = \infty$, which is equal to $\mathbf{y}_i(A)$. \square

Lemma 5.20. *Let μ be an invariant σ -finite measure of the tiling system (X, \mathbb{R}^d) , and let $U \in \mathcal{B}(X)$ be an invariant set such that $\mu(U) = 0$. Then $\mu^T(U \cap \Gamma) = 0$.*

Proof. Since U is invariant, for every patch P we have $U \cap (C_P + B_\varepsilon(\mathbf{0})) = (U \cap C_P) + B_\varepsilon(\mathbf{0})$. Thus if P is centered at $\mathbf{0}$ and ε is sufficiently small, we get

$$0 = \mu(U \cap (C_P + B_\varepsilon(\mathbf{0}))) = \mu^T(U \cap C_P) \text{vol}(B_\varepsilon(\mathbf{0}))$$

by Lemmas 7.1 and 7.2. Since $\Gamma = \bigcup_{A \in \mathcal{A}} C_A$, we deduce $\mu^T(U \cap \Gamma) = 0$. \square

Lemma 5.21. *Let μ and ν be two σ -finite ergodic invariant measures for the tiling system (X, \mathbb{R}^d) for which there exists $A \in \mathcal{A}$ such that $0 < \mu^T(C_A) = \nu^T(C_A) < \infty$ and $\mu^T|_{\mathcal{B}(C_A)} = \nu^T|_{\mathcal{B}(C_A)}$. Then $\mu = \nu$.*

Proof. Let U be the subset of X of all the tilings \mathcal{T} containing a tile equivalent to A . This set is open and invariant, and contains the set $C_A + B_\varepsilon(\mathbf{0})$, for every $\varepsilon > 0$. Thus, for ε sufficiently small we get $\mu(U), \nu(U) \geq \mu^T(C_A) \text{vol}(B_\varepsilon(\mathbf{0})) > 0$. This implies, by the ergodicity of μ and ν , that $\mu(U^c) = \nu(U^c) = 0$. Since U^c is invariant, Lemma 5.20 implies that $\mu^T(U^c \cap \Gamma) = \nu^T(U^c \cap \Gamma) = 0$. Then for every $V \in \mathcal{B}(\Gamma)$, $\mu^T(V) = \mu^T(V \cap U)$ and $\nu^T(V) = \nu^T(V \cap U)$. Let $\Lambda = \{\mathbf{v} \in \mathbb{R}^d : (\Gamma - \mathbf{v}) \cap \Gamma \neq \emptyset\}$. This set is countable because X satisfies the FPC; let $\Lambda = \{\mathbf{v}_n : n \geq 0\}$. For $V \in \mathcal{B}(\Gamma)$, we have

$$U \cap V = \bigcup_{n \geq 0} (C_A + \mathbf{v}_n) \cap V = \bigcup_{n \geq 0} V_n,$$

where $V_0 = V \cap (C_A + \mathbf{v}_0)$ and $V_n = (V \cap (C_A + \mathbf{v}_n)) \setminus (V_0 \cup \dots \cup V_{n-1})$, for $n > 0$. Then

$$\mu^T(V) = \mu^T(V \cap U) = \sum_{n \geq 0} \mu^T(V_n) \quad \text{and} \quad \nu^T(V) = \nu^T(V \cap U) = \sum_{n \geq 0} \nu^T(V_n).$$

Since for every $n \geq 0$ we have $V_n - \mathbf{v}_n \subseteq C_A$, we get

$$\mu^T(V_n) = \mu^T(V_n - \mathbf{v}_n) = \nu^T(V_n - \mathbf{v}_n) = \nu^T(V_n),$$

which implies that $\mu^T(V) = \nu^T(V)$. Theorem 7.8 implies $\mu = \nu$. \square

Theorem 5.22. *Let ω be an admissible tile substitution, which is partially recognizable and satisfies (5.3) and (2.2).*

- (i) Let $m+1 \leq i \leq l$ be such that M_i is primitive, and suppose that (5.10) holds for \mathcal{A}_i . Then the measure μ_i^T is the unique transverse measure on $\mathcal{B}(\Gamma)$ supported on $Y_i \cap \Gamma$ such that

$$\mu_i^T(C_A) = \mathbf{y}_i(A) \text{ for every } A \in \mathcal{A}.$$

Moreover, the associated invariant measure μ_i is σ -finite and ergodic.

- (ii) Suppose that (5.10) holds for all $\mathcal{A}_i \subseteq \mathcal{A}'$. Then any σ -finite ergodic measure μ with the property that $0 < \mu^T(C_A) < \infty$ for some $A \in \mathcal{A}'$, is equal to some measure μ_i up to scaling.
- (iii) If Y_i is a maximal component, then any σ -finite ergodic measure on Y_i which is positive and finite on some open set, is equal to μ_i up to scaling. (Note that (5.10) holds for maximal components by admissibility.)

Proof. (i) The uniqueness of μ_i^T follows from the fact that if ν^T is another transverse measure satisfying $\nu^T|_{\mathcal{F}_i} = \mu_i^T|_{\mathcal{F}_i}$ then $\nu^T|_{\mathcal{B}(\Gamma_i)} = \mu_i^T|_{\mathcal{B}(\Gamma_i)}$. Let μ_i be the invariant measure of $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ associated to μ_i^T . It is σ -finite and invariant by Theorem 7.8. It remains to verify that it is ergodic.

Let $U \in \mathcal{B}(X_{\mathcal{A},\omega})$ be an invariant set, that is, $U - \mathbf{v} = U$ for every $\mathbf{v} \in \mathbb{R}^d$. Since the set U is invariant the measure $\mu_U = \mu_i|_U$ is invariant. Let μ_U^T be the transverse measure on $\mathcal{B}(\Gamma)$ associated to μ_U . By Lemma 7.1, we get

$$\mu_U^T(C) = \mu_i^T(C \cap U) \text{ for every } C \in \mathcal{B}(\Gamma).$$

The measure μ_U^T verifies $\mu_U^T \leq \mu_i^T$, which ensures that μ_U^T is supported on Γ_i , and that $\mu_U(C_B) < \infty$ for every $B \in \mathcal{J}_i^c$. In particular, $\mu_U^T(C_B) = 0$ for every $B \in (\mathcal{J}_i \cup \mathcal{I}_i)^c$.

Case 1: there exists A in the class \mathcal{A}_i such that $\mu_U^T(C_A) > 0$. Then $\mu_U^T(C_B) > 0$ for every B in the class \mathcal{A}_i . Thus Lemma 5.2 implies that $\mu_U^T(C_B) = \infty$ for every $B \in \mathcal{J}_i$, and since $\mu_U^T(C_B) = 0$ for every $B \in (\mathcal{J}_i \cup \mathcal{I}_i)^c$, the vector $(\mu_U^T(C_B))_{B \in \mathcal{I}_i}$ is in the core of the restriction of M to \mathcal{I}_i . This implies that $(\mu_U^T(C_B))_{B \in \mathcal{A}} = \alpha \mathbf{y}_i$, for some $0 < \alpha \leq 1$. This vector determines $\mu_U^T(\omega^n(C_B) - \mathbf{v})$, for every $B \in \mathcal{J}_i^c$, $\mathbf{v} \in J_B^{(n)}$ and $n \geq 0$. Then $\alpha \mu_i^T|_{\mathcal{F}_i} = \mu_U^T|_{\mathcal{F}_i}$, and since μ_U^T is supported on Γ_i , we get that $\mu_U^T = \alpha \mu_i^T$. This implies that $\mu_i^T(U^c \cap \Gamma) = 0$, because of $\mu_U^T(U^c \cap \Gamma) = 0$. This shows that $\alpha = 1$ and then $\mu_U^T = \mu_i^T$. From Theorem 7.8 we obtain that $\mu_U = \mu_i$, which implies that $\mu_i(U^c) = 0$.

Case 2: If $\mu_U^T(C_A) = 0$ for every A in the class \mathcal{A}_i then $\mu_{U^c}^T(C_A) = \mu_i^T(C_A)$ for every A in the class \mathcal{A}_i . As in the previous case, but replacing U by U^c , we get that $\mu_{U^c}^T = \mu_i^T$ and $\mu_i(U) = 0$. This shows that μ_i is ergodic.

(ii) Let μ be a σ -finite ergodic measure such that $0 < \mu^T(C_A) < \infty$ for some $A \in \mathcal{A}'$. Let $i = \max\{1 \leq j \leq l : 0 < \mu^T(C_A) < \infty, A \in \mathcal{A}_j\}$. By Lemma 5.11, the vector $(\mu^T(C_A))_{A \in \mathcal{A}_i}$ is in $\text{core}(M_i)$. Then there exists $\lambda > 0$ such that for every $A \in \mathcal{A}_i$, $\mathbf{v} \in J_A^{(n)}$ and $n \geq 0$,

$$\mu^T(\omega^n(C_A) + \mathbf{v}) = \lambda \mu_i^T(\omega^n(C_A) + \mathbf{v}).$$

A standard argument shows that this equation implies that μ_T and $\lambda \mu_i^T$ coincide on each Borel set contained in $\bigcup_{A \in \mathcal{A}_i} C_A$. From Lemma 5.21 we obtain that $\mu = \lambda \mu_i$.

(iii) Let μ be an ergodic σ -finite measure on a maximal component Y_i , such that $\mu(U)$ is positive and finite for some open set $U \subseteq Y_i$. Let μ^T be the corresponding transverse measure; then $\mu^T(U \cap \Gamma)$ is positive and finite. The topology on $Y_i \cap \Gamma$ is generated by the sets C_P for $P \in \Lambda_X$ (see Section 2.7), with P containing at least one tile from \mathcal{A}_i . Decomposing C_P as a disjoint union, we can find $A \in \mathcal{A}_i$ and $n \in \mathbb{N}$ such that $\mu^T(\omega^n(C_A) - \mathbf{v})$ is positive and finite for some $\mathbf{v} \in \mathbb{R}^d$ such that $\omega^n(C_A) - \mathbf{v} \in \Gamma$. Then Lemma 5.11 implies that $\mu^T(C_A)$ is positive and finite, and we can conclude by applying part (ii). \square

Proof of Theorem B. This follows from Theorem 5.22. We only need to note that every point in a maximal component Y_i has a neighborhood of the form $(\omega^n(C_A) - \mathbf{v}) + B_\varepsilon(\mathbf{0})$, with $A \in \mathcal{A}_i$, which has positive and finite μ_i measure. \square

6. EXAMPLES AND CONCLUDING REMARKS

Example 6.1. All the tiles have the unit square as its support and are distinguished only by the labels. Let $\mathcal{A} = \{\boxed{0}, \boxed{1}, \boxed{2}\}$;

$$\boxed{0} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \quad \boxed{1} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \quad \boxed{2} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 2 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$$

The substitution matrix is $M = \begin{pmatrix} 4 & 5 & 0 \\ 5 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$. The tiling dynamical system has one minimal and one maximal component. It is easy to check that it is non-periodic. The restriction of the substitution matrix to the minimal components is $\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$. Then the unique probability measure μ is given by $\mu^T(C_2) = 0$ and

$$\mu^T(\omega^n(C_0) - \mathbf{v}) = \mu^T(\omega^n(C_1) - \mathbf{v}) = \frac{1}{2 \cdot 9^n}, \text{ for every } \mathbf{v} \in J_0^{(n)} = J_1^{(n)} \text{ and } n \geq 0.$$

This substitution satisfies (5.10); then applying Theorem 5.22 we get that every σ -finite ergodic measure μ such that $0 < \mu^T(C_2) < \infty$, is a constant multiple of the unique measure μ_2 such that μ_2^T is supported on $\bigcup_{n \geq 0} \bigcup_{\mathbf{v} \in J_2^{(n)}} (\omega^n(C_2) - \mathbf{v})$ and that verifies

$$\mu_2^T(C_2) = 1 \text{ and } \mu_2^T(C_0) = \mu_2^T(C_1) = \infty.$$

Since the restriction of the substitution matrix to \mathcal{A}' is equal to [4], we get

$$4^{-n} \mu_2^T(C_2) = \mu_2^T(\omega^n(C_2) - \mathbf{v}),$$

for every $\mathbf{v} \in J_2^{(n)}$ and $n \geq 0$.

Example 6.2 (Sierpiński carpet). All the tiles have the unit square as its support and are distinguished only by the labels. Let $\mathcal{A} = \{\boxed{0}, \boxed{1}\}$;

$$\boxed{0} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \quad \boxed{1} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

The substitution matrix is $M = \begin{pmatrix} 9 & 1 \\ 0 & 8 \end{pmatrix}$. Here the minimal component is periodic; it consists of periodic tilings with only one tile type, labeled 0. However, the “non-periodic border” condition holds (see Section 4 for details). This example, which we call the “integer Sierpiński carpet” tiling, is a generalization of the 1-dimensional symbolic substitution $0 \rightarrow 000$, $1 \rightarrow 101$, which was analyzed by A. Fisher [10]. The intersection of the transversal with the unique minimal component contains only one element $\{\mathcal{T}_0\}$. Then the transverse measure associated to the unique invariant probability measure μ_0 supported on the minimal component is the atomic measure $\mu_0^T(\{\mathcal{T}_0\}) = 1$. The measure μ_0 corresponds to the Lebesgue measure on the torus (the minimal component is conjugate to the \mathbb{R}^2 -translations on the torus). This substitution satisfies (5.10), then applying Theorem 5.22 we get that every σ -finite ergodic measure μ such that $0 < \mu^T(C_1) < \infty$, is a constant multiple of the unique measure μ_1 such that μ_1^T is supported on $\bigcup_{n \geq 0} \bigcup_{\mathbf{v} \in J_1^{(n)}} (\omega^n(C_1) - \mathbf{v})$ and verifies

$$\mu_1^T(C_1) = 1 \text{ and } \mu_1^T(C_0) = \infty.$$

Since the restriction of the substitution matrix to \mathcal{A}' is equal to [8], we get

$$8^{-n} \mu_1^T(C_1) = \mu_1^T(\omega^n(C_1) - \mathbf{v}),$$

for every $\mathbf{v} \in J_1^{(n)}$ and $n \geq 0$.

Example 6.3 (Sierpiński gasket). Consider $\mathcal{A} = \{\Delta, \nabla, \blacktriangle\}$ and the substitution given below.

$$\blacktriangle \rightarrow \blacktriangle\blacktriangle, \quad \Delta \rightarrow \Delta\Delta, \quad \nabla \rightarrow \nabla\nabla$$

The substitution matrix of ω is $M = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. The system $X_{\mathcal{A}, \omega}$ has a unique minimal component, and the submatrix of M associated to the unique minimal component is $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. This implies that the unique probability transversal measure is given by

$$\mu^T(\omega^n(C_\Delta) - \mathbf{v}) = 2^{-2n-1} \text{ and } \mu^T(\omega^n(C_\nabla) - \mathbf{u}) = 2^{-2n-1},$$

where $\mathbf{v} \in J_\Delta^{(n)}$, $\mathbf{u} \in J_\nabla^{(n)}$ and $n \geq 0$.

The tile substitution satisfies the non-periodic border condition, so it is partially recognizable (this is also easy to verify directly) and satisfies (5.10). By Theorem B, there is a

unique, up to scaling, ergodic σ -finite measure μ_\blacktriangle , for which every point containing a tile \blacktriangle has a neighborhood with positive and finite measure. For this measure we have

$$3^{-n}\mu_\blacktriangle^T(C_\blacktriangle) = \mu_\blacktriangle^T(\omega^n(C_\blacktriangle) - \mathbf{v}),$$

for every $\mathbf{v} \in J_\blacktriangle^{(n)}$ and $n \geq 0$, and $\mu_\blacktriangle^T(C_\Delta) = \mu_\blacktriangle^T(C_\nabla) = \infty$.

Example 6.4. Let $\mathcal{A} = \{\boxed{0}, \boxed{1}\}$:

$$\boxed{0} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \quad \boxed{1} \rightarrow \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}$$

This substitution is *self-affine*, rather than self-similar, and generates the integer analog of the “Bedford-McMullen carpet”, see [1, 17]. Note that here the non-periodic border condition does not hold; however, partial recognizability (i.e. every tiling containing a prototile labeled by 1 has a unique pre-image under the substitution) is easy to verify directly. Thus Theorem 5.22 applies, and we get a conclusion similar to the above examples.

6.1. Concluding remarks.

1. One can draw an (admittedly vague) analogy between substitution tiling flows and horocycle flows on manifolds of negative curvature. Moreover, the dynamics of the substitution ω is analogous to the geodesic flow. The compact manifold case corresponds to the case of minimal/primitive substitution systems, for which the horocycle flow, respectively, the tiling flow, is uniquely ergodic. The non-primitive substitutions then loosely correspond to the non-compact (but, perhaps, geometrically finite) case, where often the only “natural” invariant measure is σ -finite, see e.g. [6].
2. A. Fisher [10] obtained a “second-order ergodic theorem” for his “integer Cantor set” substitution system. Is it possible to obtain similar results for our systems? We believe that it is, at least for examples such as the “Sierpiński gasket and carpet” tilings. What about more general non-primitive substitutions? One should probably stick to the self-similar case, or else use averaging over the Følner sets $\varphi^n(B_1(\mathbf{0}))$. In general, there will be a *graph-directed Iterated Function System* associated to the tiling system. Objects analogous to the tilings, such as the “Sierpiński gasket and carpet” tilings, were considered by Strichartz, in the framework of *Reverse Iterated Function Systems* [26].
3. All our examples have tiles of very simple geometry, but there exist tile substitutions with very complicated tiles: e.g. tiles with a fractal boundary, disconnected tiles, connected tiles with disconnected interior, etc. This is known for primitive substitution tilings, see e.g. [31], and it is easy to construct a non-primitive substitution tiling using the same shapes as for a primitive one. For example, suppose we have a primitive substitution tiling $\omega : \mathcal{A} \rightarrow \mathcal{A}$, with $\omega(A) = \bigcup_{B \in \mathcal{A}} (B + D_B)$ where D_B is a finite set for every prototile B . Let $\tilde{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$, in other words, we consider “old” prototiles with an additional label. We assume that $\text{supp}(A, j) = \text{supp}(A)$ for $j = 1, 2$. Consider a non-trivial partition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and define the substitution

$\tilde{\omega} : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ by

$$\tilde{\omega}(A, 0) = \bigcup_{B \in \mathcal{A}} ((B, 0) + D_B), \quad \tilde{\omega}(A, 1) = \bigcup_{B \in \mathcal{A}_1} ((B, 0) + D_B) \cup \bigcup_{B \in \mathcal{A}_2} ((B, 1) + D_B).$$

This is a non-primitive tile substitution, and one can refine this construction to satisfy any additional properties, such as admissibility, non-periodic border, etc.

7. APPENDIX: INVARIANT MEASURES VERSUS TRANSVERSE MEASURES.

We use the notation and terminology from Section 2.7. Recall that a *transverse measure* on $\mathcal{B}(\Gamma)$ is a measure $\mu : \mathcal{B}(\Gamma) \rightarrow \overline{\mathbb{R}}_+$ such that $\mu(A) = \mu(A - \mathbf{v})$, for every $A \subseteq \mathcal{B}(\Gamma)$ and $\mathbf{v} \in \mathbb{R}^d$ for which $A - \mathbf{v} \subseteq \Gamma$ (see [2, Definition 5.1] for a definition of transverse measure in the context of laminations). Recall that $\eta > 0$ is such that the closure of $B_\eta(\mathbf{0})$ is contained in the interior of every prototile. Observe that

$$(7.1) \quad \mathcal{T} \in \Gamma, \quad \mathcal{T} + \mathbf{v} \in \Gamma, \quad \mathbf{v} \in B_{2\eta}(\mathbf{0}) \Rightarrow \mathbf{v} = \mathbf{0}.$$

We write $X := X_{\mathcal{A}, \omega}$ to simplify the notation.

7.1. From invariant measures to transverse measures.

Lemma 7.1. *Let μ be an invariant measure of (X, \mathbb{R}^d) . For every $U \in \mathcal{B}(\Gamma)$, there exists $\mu^T(U) \in \overline{\mathbb{R}}_+$ such that for every open set Θ contained in the ball $B_\eta(\mathbf{0})$, we have*

$$\frac{\mu(U + \Theta)}{\text{vol}(\Theta)} = \mu^T(U).$$

Proof. Fix $U \subseteq \mathcal{B}(\Gamma)$. Observe that $\mu_U : E \mapsto \mu(U + E)$ is a Borel measure on the ball $B_\eta(\mathbf{0})$. This follows from (7.1), which implies

$$E_1, E_2 \subseteq B_\eta(\mathbf{0}), \quad E_1 \cap E_2 = \emptyset \Rightarrow (U + E_1) \cap (U + E_2) = \emptyset.$$

Moreover, by the invariance of μ we have

$$E \subseteq B_\eta(\mathbf{0}), \quad E - \mathbf{v} \subseteq B_\eta(\mathbf{0}) \Rightarrow \mu(E - \mathbf{v}) = \mu(E).$$

It is easy to see that if for some open $\Theta_1 \subseteq B_\eta(\mathbf{0})$ we have $\mu(U + \Theta_1) = 0$, then $\mu(U + \Theta) = 0$ for all open subsets of $B_\eta(\mathbf{0})$ and we can set $\mu^T(U) = 0$. Similarly, if for some open $\Theta_1 \subseteq B_\eta(\mathbf{0})$ we have $\mu(U + \Theta_1) = \infty$, then $\mu(U + \Theta) = \infty$ for all open subsets of $B_\eta(\mathbf{0})$ and we can set $\mu^T(U) = \infty$. So we can suppose that μ_U is positive and finite on open subsets of $B_\eta(\mathbf{0})$. Consider the restriction of μ_U to a cube $\prod_{i=1}^d [a_i, a_i + h]$ contained in $B_\eta(\mathbf{0})$ and extend it to \mathbb{R}^d by periodicity, i.e. let

$$\nu_U(E) = \sum_{\mathbf{x} \in \mathbb{Z}^d} \mu \left(U + \left(E \cap \left(\prod_{i=1}^d [a_i, a_i + h] + h\mathbf{x} \right) \right) \right).$$

This is a Borel measure on \mathbb{R}^d which is translation-invariant and positive and finite on open subsets. It follows that $\nu_U(E) = c_U \text{vol}(E)$ and we can set $\mu^T(U) = c_U$. \square

Lemma 7.2. *Let $\mu^T : \mathcal{B}(\Gamma) \rightarrow \overline{\mathbb{R}}_+$ be the function obtained in Lemma 7.1. Then μ^T is a transverse measure.*

Proof. It is clear that μ^T is a measure. Indeed, if $(U_n)_{n \in \mathbb{N}}$ is a collection of disjoint sets in $\mathcal{B}(\Gamma)$, and $\varepsilon > 0$ is small enough, then the sets $(U_n + B_\varepsilon(\mathbf{0}))_{n \in \mathbb{N}}$ are disjoint. It follows from the definition of μ^T that $\mu^T(\bigcup_{n \in \mathbb{N}} U_n) = \sum_{n \in \mathbb{N}} \mu^T(U_n)$. If $U \in \mathcal{B}(\Gamma)$ and $\mathbf{v} \in \mathbb{R}^d$ is such that $U - \mathbf{v} \subseteq \Gamma$, then for $\varepsilon > 0$ we have $\mu(U - \mathbf{v} + B_\varepsilon(\mathbf{0})) = \mu(U - B_\varepsilon(\mathbf{0}))$, which implies that μ^T is transverse. \square

Definition 7.3. Let μ be an invariant measure of (X, \mathbb{R}^d) . We denote by μ^T the transverse measure associated to μ .

7.2. From transverse measures to invariant measures. Let ν be a σ -finite transverse measure on Γ . We write λ_d for the Lebesgue measure on \mathbb{R}^d , and $\nu \otimes \lambda_d$ for the product measure on $\Gamma \times \mathbb{R}^d$.

For every $\mathbf{w}, \mathbf{v} \in \mathbb{R}^d$, we define $\psi^{(\mathbf{w}, \mathbf{v})} : X \times \mathbb{R}^d \rightarrow X \times \mathbb{R}^d$ by

$$\psi^{(\mathbf{w}, \mathbf{v})}(\mathcal{T}, \mathbf{u}) = (\mathcal{T} - \mathbf{w}, \mathbf{u} - \mathbf{v}), \text{ for every } \mathcal{T} \in X \text{ and } \mathbf{u} \in \mathbb{R}^d.$$

This function is a homeomorphism (with respect to the product topology).

Lemma 7.4. Let U be a Borel set in $\Gamma \times \mathbb{R}^d$. Then

$$\nu \otimes \lambda_d(\psi^{(\mathbf{w}, \mathbf{v})}(U)) = \nu \otimes \lambda_d(U),$$

for every $(\mathbf{w}, \mathbf{v}) \in \mathbb{R}^{2d}$ such that $\psi^{(\mathbf{w}, \mathbf{v})}(U) \subseteq \Gamma \times \mathbb{R}^d$.

Proof. For every U in $X \times \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$ we set

$$(U)^1(\mathbf{x}) = \{\mathcal{T} \in X : (\mathcal{T}, \mathbf{x}) \in U\}.$$

Let U be a Borel set in $\Gamma \times \mathbb{R}^d$ and let $(\mathbf{w}, \mathbf{v}) \in \mathbb{R}^d$ be such that $\psi^{(\mathbf{w}, \mathbf{v})}(U) \subseteq \Gamma \times \mathbb{R}^d$. We have

$$\begin{aligned} \nu \otimes \lambda_d(\psi^{(\mathbf{w}, \mathbf{v})}(U)) &= \int \nu((\psi^{(\mathbf{w}, \mathbf{v})}(U))^1(\mathbf{x})) d\lambda_d(\mathbf{x}) \\ &= \int \nu(U^1(\mathbf{x} + \mathbf{v}) - \mathbf{w}) d\lambda_d(\mathbf{x}) \end{aligned}$$

Since ν is transverse and $U^1(\mathbf{y}) - \mathbf{w} \subseteq \Gamma$, for every $\mathbf{y} \in \mathbb{R}^d$, we get

$$\int \nu(U^1(\mathbf{x} + \mathbf{v}) - \mathbf{w}) d\lambda_d(\mathbf{x}) = \int \nu(U^1(\mathbf{x} + \mathbf{v})) d\lambda_d(\mathbf{x}).$$

The invariance under translations of the Lebesgue measure implies that

$$\int \nu(U^1(\mathbf{x} + \mathbf{v})) d\lambda_d(\mathbf{x}) = \int \nu(U^1(\mathbf{x})) d\lambda_d(\mathbf{x}) = \nu \otimes \lambda_d(U).$$

\square

Since X verifies FPC, X is a finite union of sets $C_P + \Theta$, with $P \in \Lambda_X$ and Θ open in \mathbb{R}^d with diameter smaller than η . Namely, $X = \bigcup_{i=1}^n U_i$, where $U_i = C_{P_i} + \Theta_i$, for every $1 \leq i \leq n$.

From (7.1), for each $1 \leq i \leq n$ the function $h_i : U_i \rightarrow C_i \times \Theta_i$ given by $h_i(\mathcal{T} + \mathbf{v}) = (\mathcal{T}, \mathbf{v})$ is well-defined. Moreover, h_i is a homeomorphism.

For every $1 \leq i \leq n$, let $\mathbf{a}_i \in \mathbb{R}^d$ be a vector such that $\mathbf{0} \in \Theta_i - \mathbf{a}_i$ and $\Theta_i - \mathbf{a}_i$ is contained in the ball $B_\eta(\mathbf{0})$. Since for every $R > 0$, the set C_{P_i} is a finite and disjoint union of sets C_P , with $P \in \Lambda_X$ whose support contains the ball $B_R(\mathbf{0})$, we can assume that the support of C_{P_i} contains the vectors $\mathbf{a}_k - \mathbf{a}_j$, for every $1 \leq k, j \leq n$. This implies that for every $1 \leq i, j \leq n$, there exist $\mathbf{a}_{i,j}, \mathbf{b}_{i,j} \in \mathbb{R}^d$ such that

$$(7.2) \quad h_j \circ h_i^{-1}(\mathcal{T}, \mathbf{v}) = (\mathcal{T} + \mathbf{a}_{i,j}, \mathbf{v} + \mathbf{b}_{i,j}), \text{ for every } \mathcal{T} \in C_{P_i}, \mathbf{v} \in \Theta_i.$$

Equation (7.2) implies that X has a *d-lamination structure* (see [2] for details). The collection $\{U_i, h_i\}_{i=1}^n$ is called an *atlas* of X .

For every $1 \leq i \leq n$ define $\mu_i : \mathcal{B}(U_i) \rightarrow \overline{\mathbb{R}}_+$ by $\mu_i(U) = \nu \otimes \lambda_d(h_i(U))$. It is clear that μ_i is a measure.

We define $\tilde{U}_1 = U_1$ and $\tilde{U}_i = U_i \setminus (\bigcup_{j=1}^{i-1} U_j)$, for every $2 \leq i \leq n$. The function $\mu = \sum_{i=1}^n \mu_i|_{\tilde{U}_i}$ is a measure on $\mathcal{B}(X)$. Since $\nu \otimes \lambda_d$ is σ -finite, μ is σ -finite too. We will show that μ is invariant and that $\mu^T = \nu$.

Lemma 7.5. *The measure μ does not depend on the atlas.*

Proof. Since X verifies FPC, the set $\Lambda = \{\mathbf{v} \in \mathbb{R}^d : (\Gamma - \mathbf{v}) \cap \Gamma \neq \emptyset\}$ is countable. Let $\Lambda = \{\mathbf{v}_n : n \in \mathbb{N}\}$.

Let $\{V_i, f_i\}_{i=1}^m$ be another atlas of X . For every $1 \leq i \leq m$, let $\tilde{\mu}_i$ be the measure on V_i defined as $\tilde{\mu}_i = (\nu \otimes \lambda_d) \circ f_i$. We set $\tilde{V}_1 = V_1$ and $\tilde{V}_i = V_i \setminus (\bigcup_{j=1}^{i-1} V_j)$, for every $2 \leq i \leq m$. Denote by $\tilde{\mu}$ the measure on X defined by $\tilde{\mu} = \sum_{i=1}^m \tilde{\mu}_i|_{\tilde{V}_i}$.

Let \mathcal{T} be a tiling in $U_i \cap V_j$, for some $1 \leq i \leq n$ and $1 \leq j \leq m$. If $h_i(\mathcal{T}) = (\mathcal{T}_i, \mathbf{v}_i)$ then there exists $\mathbf{v}(\mathcal{T}) \in \Lambda$ such that $f_j(\mathcal{T}) = (\mathcal{T}_i + \mathbf{v}(\mathcal{T}), \mathbf{v}_i - \mathbf{v}(\mathcal{T}))$. Since $\mathbf{v}(\mathcal{T})$ is in Λ , there exists $n \in \mathbb{N}$ such that $\mathbf{v}(\mathcal{T}) = \mathbf{v}_n$. Thus if U is a Borel set in $U_i \cap V_j$, it can be written as

$$U = \bigcup_{n \in \mathbb{N}} U_n, \text{ where } U_n = \{\mathcal{T} \in U : f_j(\mathcal{T}) = \psi^{\mathbf{v}_n}(h_i(\mathcal{T}))\},$$

where $\psi^{\mathbf{v}_n}$ abbreviates $\psi^{(\mathbf{v}_n, -\mathbf{v}_n)}$. The sets U_n are disjoint and measurable.

From Lemma 7.4, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \mu_i(U_n) &= \nu \otimes \lambda_d(h_i(U_n)) \\ &= \nu \otimes \lambda_d(\psi^{\mathbf{v}_n}(h_i(U_n))) \\ &= \nu \otimes \lambda_d(f_j(U_n)) \\ &= \tilde{\mu}_j(U_n), \end{aligned}$$

which implies that $\mu_i(U) = \tilde{\mu}_j(U)$, and hence $\mu = \tilde{\mu}$. \square

Remark 7.6. From the proof of Lemma 7.5 we also deduce that $\mu(U) = \mu_i(U)$, for every Borel set $U \subseteq U_i$.

Lemma 7.7. *The measure μ is invariant and $\mu^T = \nu$.*

Proof. Let $\mathbf{v} \in \mathbb{R}^d$. For every $1 \leq i \leq n$, let $V_i = U_i - \mathbf{v}$ and $f_i : V_i \rightarrow C_{P_i} \times (\Theta_i - \mathbf{v})$ be defined as $f_i(\mathcal{T}) = \psi^{(\mathbf{0}, \mathbf{v})}(h_i(\mathcal{T} + \mathbf{v}))$. The collection $\{V_i, f_i\}$ is an atlas of X .

Let U be a Borel set in U_i . We have $U - \mathbf{v} \subseteq V_i$. Then from Lemma 7.4 and Lemma 7.5 we get

$$\begin{aligned}\mu(U - \mathbf{v}) &= \nu \otimes \lambda_d(f_i(U - \mathbf{v})) \\ &= \nu \otimes \lambda_d(\psi^{(\mathbf{0}, \mathbf{v})}(h_i(U))) \\ &= \nu \otimes \lambda_d(h_i(U)) \\ &= \mu_i(U) \\ &= \mu(U).\end{aligned}$$

This shows that μ is invariant.

Let $C \in \mathcal{B}(\Gamma)$ and $\Theta \subseteq B_\eta(\mathbf{0})$ an open set. We can assume that $C + \Theta$ is the disjoint union of sets $C_i + \Theta$, where $C_i \subseteq C_{P_i}$ and $\Theta \subseteq \Theta_i$, for every $1 \leq i \leq n$. We have

$$\mu(C + \Theta) = \nu^T \otimes \lambda_d(C \times \Theta) = \nu(C) \lambda_d(\Theta).$$

Hence $\mu(C + \Theta) = \nu(C) \lambda_d(\Theta)$, which implies that $\mu^T = \nu$. \square

Theorem 7.8. *There is a linear one-to-one correspondence between the set of σ -finite invariant measures and the set of σ -finite transverse measures of (X, \mathbb{R}^d) .*

Proof. Lemma 7.7 shows that the function that associates to every invariant measure μ its transverse measure μ^T is onto.

Let ν be a transverse measure and let μ be an invariant measure such that $\mu^T = \nu$. Let $\{U_i, h_i\}_{i=1}^n$ be an atlas of X . The measure $\mu \circ h_i^{-1}$ is defined on $h_i(U_i)$ and verifies

$$\mu \circ h_i^{-1}(C \times \Theta) = \nu \otimes \lambda_d(C \times \Theta),$$

for every $C \times \Theta \in (\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)) \cap h_i(U_i)$. The uniqueness of the product measure implies that μ is the invariant measure of Lemma 7.7. Therefore, the function that associates to every invariant measure μ its transverse measure μ^T is one-to-one. The linearity is clear. \square

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